SKETCH OF SOLUTIONS TO THE MIDTERM EXAM

Problem 0.1. Let D_6 be the dihedral group with 12 elements. Give two *different non-trivial* actions of D_6 on the sphere by isometries (i.e. such that the quotients give two different orbifolds).

Proof. (Sketch) Recall that $D_6 =< a, b \mid a^6 = b^2 = (ab)^2 = 1 >$. Let S^2 be the unit sphere in $\mathbb{R}^3$. Let a act on S^2 by counterclockwise rotation of order 6 around z-axis. There are two different actions of b on the sphere. First, let b act by rotation by π around x-axis (orientation-preserving). For the second action, let b act by reflection about xy-plane (orientation-reversing).

To get full credit, you should verify that these actions are indeed different by showing that the quotient orbifolds are different (for example, show that they have different Euler characteristics). $\square$

Problem 0.2. Give an example of a group G that acts discretely by isometries on the plane, and which is torsion-free, but is not a group of translations. Give an example of a group G that acts discretely by isometries on 3-dimensional Euclidean space, and which is torsion-free and infinite, but in which no non-trivial element is a translation.

Proof. (Sketch) Let a be the translation by 1 unit along the y-axis (i.e. $(x, y) \mapsto (x, y + 1)$) and b be the glide reflection along x-axis (i.e. $(x, y) \mapsto (x + 1, -y)$. Consider the group G genereated by a and b. Verify that this group is torsion-free and acts discretely by isometries.

Let c be a rotation by an irrational multiple of π followed by a translation, and let G be the group generated by c. Verify that G acts discretely by isometries and is torsion-free and infinite, but only the trivial element is a translation. $\square$

Problem 0.3. Suppose Σ is a closed surface for which there is a nontrivial finite group G that acts on Σ in such a way that the quotient Σ/G is homeomorphic to Σ. What does this imply about the Euler characteristic of Σ? Give an example. Give an example where Σ is an orbiford which is not a surface.

Proof. (Sketch) Recall that $\chi(\Sigma/G) = \frac{\chi(\Sigma)}{|G|}$. Since Σ/G is homeomorphic to Σ, $\chi(\Sigma/G) = \chi(\Sigma)$. Hence $\chi(\Sigma) = 0$ since G is nontrivial.

Manifold example: Let Σ be a torus with an action by finite-order rotation along the longtitude. Note that the quotient is homeomorphic to a torus.

Orbifold example: Let Σ be a square with mirror egdes and 4 order-2 corner reflectors. Let $G = \mathbb{Z}/2\mathbb{Z}$ act by reflection about the vertical line connecting the midpoints of horizontal edges. You need to verify that the quotient is homeomorphic to Σ. $\square$

Problem 0.4. A buckyball is a tiling of the sphere by pentagons and hexagons, meeting 3 around every vertex. Thus a soccerball is an example of a buckyball. How many pentagons are there in a buckyball?

Proof. Suppose there are f_5 pentagons and f_6 hexagons. Then the Euler characteristic of a sphere is

$$2 = \frac{5f_5 + 6f_6}{3} - \frac{5f_5 + 6f_6}{2} + f_5 + f_6.$$

Hence, $f_5 = 12$. $\square$