BANACH-TARSKI PARADOX

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Free group. Let $F = \langle a, b\rangle$ be free on elements a and b. We denote a^{-1} by A and b^{-1} by B for simplicity. The group F acts on itself by left multiplication. Elements of F are uniquely represented by *reduced* words in a, A, b, B; i.e. such that a is never adjacent to A, and b is never adjacent to B.

We may partition F into the following four subsets:

1. P_1 consisting of all words starting with A but *not* equal to a positive power of A;
2. P_2 consisting of all non-negative powers of A, together with all words starting with a;
3. P_3 consisting of all words starting with b; and
4. P_4 consisting of all words starting with B.

These four sets are disjoint. Moreover, aP_1 is equal to the complement of P_2, and bP_4 is equal to the complement of P_3. Thus the action of F on itself is *paradoxical*, in the sense that F can be decomposed into the 4 disjoint sets P_i, and then we can make two copies of F from translates of the pieces: $F = aP_1 \cup P_2$ and $F = P_3 \cup bP_4$.

Hyperbolic geometry. Let Γ be a discrete cocompact subgroup of the group of isometries of the hyperbolic plane $\mathbb{H}$. Then Γ contains a copy of F acting freely. This is proved by Klein's "ping-pong" lemma. The key is to find two hyperbolic elements α and β with disjoint fixed points $p^\pm$ and $q^\pm$ respectively, and then to replace α by big powers $a := \alpha^N$ and $b := \beta^N$ so that there are neighborhoods $U^\pm$ of $p^\pm$ and $V^\pm$ of $q^\pm$ so that

1. a takes the complement of U^- into U^+;
2. A takes the complement of U^+ into U^-;
3. b takes the complement of V^- into V^+; and
4. B takes the complement of V^+ into V^-.

Once we know that a and b generate a copy of F acting freely on $\mathbb{H}$ we can produce a paradoxical decomposition for Γ acting on $\mathbb{H}$.

Let X be a fundamental domain for the action (i.e. a set of orbit representatives, so that for each point p in $\mathbb{H}$ there is exactly one point x_p in X so that $gp = x_p$ for some unique $g \in F$). We can then define a decomposition of $\mathbb{H}$ into the disjoint subsets $X_j = P_jX$ for P_j the subset of F defined above.

Then just as before, $\mathbb{H}$ is the disjoint union of X_1, X_2, X_3 and X_4, and we have decompositions $\mathbb{H} = aX_1 \cup X_2$ and $\mathbb{H} = X_3 \cup bX_4$.

Free subgroups of rotations. We use number theory to construct a free subgroup of the group of rotations of the 2-sphere.

First consider the quadratic form $Q^- := x^2 + y^2 - \sqrt{2}z^2$ on $\mathbb{R}^3$. The matrices

$$R := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} -1-\sqrt{2} & 2+\sqrt{2} & 2+2\sqrt{2} \\ -2-\sqrt{2} & 1+\sqrt{2} & 2+2\sqrt{2} \\ -2-\sqrt{2} & 2+\sqrt{2} & 3+2\sqrt{2} \end{pmatrix}, \quad T := \begin{pmatrix} 0 & 3-2\sqrt{2} & 2-2\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 2-2\sqrt{2} & 3-2\sqrt{2} \end{pmatrix}$$

preserve the form Q^-, and generate a group Γ^- which can be conjugated into $\text{Isom}^+(\mathbb{H})$ by multiplying the last column of each matrix by $2^{-1/4}$ and the last row by $2^{1/4}$ (so that the bottom rightmost entry is unchanged). The image is a discrete cocompact subgroup of $\text{Isom}^+(\mathbb{H})$ and therefore contains a copy of the free group F.

Next consider the quadratic form $Q^+ := x^2 + y^2 + \sqrt{2}z^2$ on $\mathbb{R}^3$. The matrices in Γ^- have entries in the ring $\mathbb{Z}[\sqrt{2}]$. There is a *Galois automorphism* σ of this ring, defined by

$$\sigma : a + b\sqrt{2} \to a - b\sqrt{2}$$

for $a, b \in \mathbb{Z}$. This automorphism takes the group Γ^- to a group Γ^+ of matrices which preserve the form Q^+, and which can be conjugated into $\text{Isom}^+(S^2)$ by multiplying the last column of each matrix by $2^{-1/4}$ and the last row by $2^{1/4}$. Thus $\text{Isom}^+(S^2)$ contains a copy of the free group F.

Fixed points. A nontrivial element of $\text{Isom}^+(S^2)$ fixes exactly two points, so there are countably many points Y in S^2 fixed by some element of our free group F. Let θ be an irrational rotation with the property that the translates $\theta^n Y$ are disjoint from each other, for all non-negative integers n. Define $Z := Y \cup \theta Y \cup \theta^2 Y \cup \cdots$. Then $\theta Z = Z - Y$ so we can partition S^2 into the sets $S^2 - Z$ and Z, apply θ to Z, and observe that $S^2 - Z \cup \theta Z = S^2 - Y$.

But F acts freely on $S^2 - Y$, so we can choose a set of orbit representatives X for the action exactly as before, define $X_j = P_j X$, and observe that $S^2 - Y$ can be partitioned into X_1, X_2, X_3, X_4 such that $S^2 - Y = aX_1 \cup X_2$ and $S^2 - Y = X_3 \cup bX_4$. Then decompose each of these $S^2 - Y$s into $S^2 - Z$ and $Z - Y$, and apply θ^{-1} to $Z - Y$ to obtain Z, and thereby obtain a paradoxical decomposition for the action of $\text{Isom}^+(S^2)$ on S^2.

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