RIEMANNIAN GEOMETRY, SPRING 2013, MIDTERM

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This midterm exam was given out in class on Friday, May 3rd.

Problem 1. The *helicoid* is the surface in $\mathbb{E}^3$ given in parametric form by the equations

$$x = \rho\cos(\theta),\ y = \rho\sin(\theta),\ z = \theta$$

for $\rho, \theta \in (-\infty, \infty)$. Show that the second fundamental form has zero trace at every point.

Problem 2. Let $M = S^2 \times S^2$ with the product metric $g := g_1 \oplus g_2$ where g_1 and g_2 are metrics on the two sphere factors making them isometric to the round spheres in $\mathbb{E}^3$ of radius 1 and 2 respectively. Let $T \subset M$ be a torus, obtained by taking the product of the equators in the two spheres. Show that T (with its intrinsic metric) is flat, and that it is totally geodesic in M (i.e. that geodesics on T are geodesics in M).

Problem 3. Give an example of two *different* Riemannian manifolds diffeomorphic to S^2 (i.e. two Riemannian metrics on S^2 which are not isometric as Riemannian manifolds), and a diffeomorphism f from one sphere to the other such that for every point p in the first sphere, the sectional curvature at p is equal to the sectional curvature of the second sphere at $f(p)$. (Bonus question: give an example where the sectional curvatures are strictly positive)

Problem 4. Let S be a Riemannian surface, and $p \in S$ be a point. Let $B_r(0)$ be the ball of radius r in T_pS centered at 0, and let $\gamma_r = \exp_p(\partial B_r(0))$. Show that the sectional curvature $K(p)$ of S at p satisfies

$$K(p) = \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r - \text{length}(\gamma_r)}{r^3}$$

Problem 5. Give an example of a Riemannian manifold whose sectional curvature is everywhere strictly positive, and which is complete but noncompact.

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