RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 8

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 31st.

Problem 1. Recall that the *volume form* on an n-dimensional vector space V with an inner product is a choice of element of $\Lambda^n V$ of length 1. Let M be an oriented Riemannian manifold, so that the Riemannian metric determines a volume form pointwise on M, and therefore (globally) an n-form ω called "the volume form on M". Let $x_1, \cdots, x_n$ be local coordinates on M, and let the metric on these local coordinates be given by $g_{ij} := \langle \partial_i, \partial_j \rangle$. Show that the volume form can be expressed in terms of these coordinates as $\sqrt{\det(g)} dx^1 \wedge \cdots dx^n$ where $\det(g)$ is the function which, at each point p, is equal to the determinant of the $n \times n$ matrix with ij entry $g_{ij}(p)$.

Problem 2. Let M be an oriented Riemannian manifold. Recall that the *divergence* of a vector field X is the function $\text{div}(X) := *d*X^\flat$ where $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the "musical isomorphism" identifying vectors and 1-forms pointwise by the way they pair with vectors.

Let D be a smooth compact codimension 0 submanifold of M with smooth boundary ∂D (which is a smooth, closed, codimension 1 submanifold of M). Let ν be the *outward* normal unit vector field on ∂D; i.e. the vector field pointing out of D and into $M - D$. Prove the *divergence theorem*:

$$\int_D \text{div}(X) d\text{vol} = \int_{\partial D} \langle X, \nu \rangle d\text{area}$$

where $d$vol and $d$area denote the (oriented) volume forms on D and ∂D respectively.

Problem 3. Let M be a closed, oriented 3-manifold. Let ξ be a smooth distribution of 2-planes on M (i.e. $\xi(p)$ is a 2-dimensional subspace of T_pM for all p). We say ξ is a *contact structure* if locally there is a 1-form α (called a *contact form*) such that $\ker(\alpha) = \xi$, and $\alpha \wedge d\alpha \neq 0$. If M is oriented, we call ξ a *positive* contact structure if $\alpha \wedge d\alpha$ is a positively oriented volume form (locally).

Suppose ξ is a positive contanct structure, and suppose there is a global 1-form α with $\ker(\alpha) = \xi$.
(i): Show there is a unique vector field $X \in \mathfrak{X}(M)$ with X contained in $\ker(d\alpha)$ (i.e. $\iota_X d\alpha = 0$) and $\alpha(X) = 1$. Such an X is called the *Reeb vector field* associated to the contact form α.
(ii): Show that the flow generated by X preserves the "volume form" $\alpha \wedge d\alpha$.
(iii): Let α be a contact form and X the Reeb flow. Show that there is a Riemannian metric for which $X^\flat = \alpha$. Furthermore, show that for such a Riemannian metric, the flowlines are geodesics.
(iv): On an oriented Riemannian 3-manifold, the *curl* of a vector field X is defined by $\text{curl}(X) := (*dX^\flat)^\sharp$ where $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ is the inverse of $\flat$. A vector field X for which there is a smooth function f with $\text{curl}(X) = fX$ is called a *Beltrami field* in hydrodynamics. Suppose that X is a nowhere vanishing Beltrami field, and suppose further that the function f as above is nowhere vanishing. Show in this case that X may be rescaled (i.e. multiplied pointwise by a smooth nonvanishing function) to be the Reeb flow of some contact form.

Problem 4. Let M be a closed, oriented Riemannian manifold. Recall that the Laplacian Δ on p-forms is defined by $\Delta = \delta d + d\delta$, so that the Laplacian on functions is just δd. Suppose α is a p-form on M which can be expressed locally in the form $\alpha = \sum_I f_I dx^I$ for smooth functions f_I and multi-indices $I = i_1, i_2, \cdots, i_p$ with $i_1 < i_2 < \cdots < i_p$. Show that the leading order part of Δ has the form

$$\Delta\alpha = \sum_I (\Delta f_I) dx^I + \text{lower order terms}$$

Problem 5. Let M be a closed, oriented manifold, and let α be a *nondegenerate* closed 1-form; i.e. a 1-form with $d\alpha = 0$ and $\alpha \neq 0$ everywhere. Prove *Tischler's theorem* that there is a submersion $M \to S^1$; thus M has the structure of a fiber bundle over S^1. (Hint: pick a Riemannian metric and use the Hodge theorem)

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