RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 7

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 24th.

Problem 1. Let G be a Lie group, and H a closed subgroup. Prove that the space G/H is complete in any G-invariant metric (where G acts on the left in the obvious way).

Problem 2. Let G be a Lie group. Show that there is some open neighborhood U of the identity element e so that any subgroup Γ of G (closed or not, discrete or not) contained in U is equal to e (this is expressed by saying that a Lie group has *no small subgroups*). (Bonus question: give an example of a locally compact topological group which fails to have this property)

Problem 3. For any n let $\mathrm{SL}(n,\mathbb{R})$ denote the group of $n\times n$ real matrices with determinant 1. Embed $\mathrm{SL}(n-1,\mathbb{R})$ as a subgroup of $\mathrm{SL}(n,\mathbb{R})$ by the homomorphism $M \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & M \end{smallmatrix}\right)$. The group $\mathrm{SL}(n,\mathbb{R})$ acts on the homogeneous space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n-1,\mathbb{R})$ by multiplication on the left. Show that for $n \ge 3$ this homogeneous space admits no (left)-invariant metric.

Problem 4. The Lie group Nil (also sometimes called the *Heisenberg group*) is the group

$$\mathrm{Nil} = \left\{ 3\times 3 \text{ real matrices of the form } \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

(i): Find a basis for the Lie algebra and compute the adjoint action in those coordinates.
(ii): Give an explicit closed formula for the exponential map and show that it is a diffeomorphism from the Lie algebra to the group.
(iii): Define vector fields in $\mathfrak{X}(\mathbb{R}^3)$ by the formulae

$$X := \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad Y := \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \quad Z := \frac{\partial}{\partial z}$$

Show that $[X,Y]=Z$ and $[X,Z]=[Y,Z]=0$. Use this to construct an identification of Nil with $\mathbb{R}^3$.
(iv): Let θ denote the 1-form $\theta := dz - \frac{1}{2}(xdy - ydx)$. The 2-plane field $\xi = \ker(\theta)$ is a distribution spanned locally by X and Y. Show that for any points p and q there is a smooth path γ from p to q with $\theta(\gamma')=0$. In fact, show that for *any* continuous path δ from p to q there is a smooth path γ which is C^0 close to δ (i.e. arbitrarily close in the C^0 topology), satisfies $\theta(\gamma')=0$, and runs from p to q.
(v): Forgetting the z coordinate defines a projection $\mathbb{R}^3 \to \mathbb{R}^2$, where we think of the image as the x–y plane. Show that if $\gamma : [0,1] \to \mathbb{R}^2$ is a smooth curve in the x–y plane, and $p \in \mathbb{R}^3$ is any point that projects to $\gamma(0)$, there is a unique smooth curve $\tilde{\gamma} : [0,1] \to \mathbb{R}^3$ with $\theta(\tilde{\gamma}')=0$ which projects to γ and starts at p. If $\gamma(0)=\gamma(1)$ (so that the image is a smooth, immersed circle in $\mathbb{R}^2$) show that the difference $z(\tilde{\gamma}(1)) - z(\tilde{\gamma}(0))$ is equal to half the *algebraic area* enclosed by γ.

Note: for a smooth map $\gamma : S^1 \to \mathbb{R}^2$, the algebraic area enclosed by γ is defined as follows. For each point $p \in \mathbb{R}^2 - \gamma(S^1)$, join p to infinity by a smooth ray δ_p, and define $\mathrm{wind}(\gamma,p)$ to be the algebraic intersection number of δ_p with $\gamma(S^1)$. Then the algebraic area enclosed by γ is just $\int_{\mathbb{R}^2} \mathrm{wind}(\gamma,p) d\mathrm{area}(p)$.

Problem 5. Recall that $\mathrm{Sp}(2n,\mathbb{R})$ is the group of $2n\times 2n$ real matrices preserving the standard symplectic form $\omega := \sum_i dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ under their linear action. We identify $\mathbb{R}^{2n} = \mathbb{C}^n$ in the usual way, and let $\mathrm{U}(n)$ denote the group of $n\times n$ complex matrices preserving the standard Hermitian form on $\mathbb{C}^n$.
(i): Show that with identifications as above, $\mathrm{U}(n)$ is a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$.

(ii): The quotient space $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$ is called the *Siegel upper half-space* $\mathfrak{H}_n$, generalizing the usual upper half-plane model of hyperbolic space when $n = 1$. Show that we can identify $\mathfrak{H}_n$ with the set of complex $n \times n$ matrices Z which are symmetric and whose imaginary part is positive-definite, where the action of $\mathrm{Sp}(2n,\mathbb{R})$ is as follows: if we write $M \in \mathrm{Sp}(2n,\mathbb{R})$ in $n \times n$ blocks as $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, then

$$M \cdot Z = (AZ + B)(CZ + D)^{-1}$$

(iii): Show that $\mathfrak{H}_n$ admits a Riemannian metric invariant under the (left) action of $\mathrm{Sp}(2n,\mathbb{R})$. (Bonus question: show that this metric is non-positively curved but not strictly negatively curved)

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