RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 6

DANNY CALEGARI

Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 17th.

Problem 1. Give an example of a Riemannian metric on a sphere containing a non-periodic infinite geodesic which is not dense (i.e. its closure is not the entire sphere).

Problem 2. A *submersion* is a differentiable map between smooth manifolds $\pi : M^{n+k} \to N^n$ such that at each point $d\pi$ has rank n (i.e. it is surjective on tangent spaces). It follows that for each $p \in N$ the preimage $\pi^{-1}(p)$ is a smooth k-dimensional submanifold of M. Let V_q denote the tangent space to $\pi^{-1}(p)$ at some point $q \in \pi^{-1}(p)$. If M is a Riemannian metric, set H_q to be the subspace of T_qM perpendicular to V_q, so that $TM = V \oplus H$.

The map π is said to be a *Riemannian submersion* if $d\pi|H$ is an isometry. Suppose $\pi : M \to N$ is a Riemannian submersion.

(i): Show for each vector field X on N there is a unique vector field $\bar{X}$ on M so that $\bar{X}$ is everywhere contained in H, and $d\pi(\bar{X}) = X$.

(ii): If X, Y, Z are vector fields on N, show that

$$\langle [\bar{X}, \bar{Y}], \bar{Z} \rangle = \langle [X, Y], Z \rangle$$

Conclude that

$$\langle \bar{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle = \langle \nabla_X Y, Z \rangle$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on M, and ∇ the Levi-Civita connection on N.

(iii): If X and Y are vector fields on N and T is vertical (i.e. is a section of V), show

$$\langle [\bar{X}, T], \bar{Y} \rangle = 0$$

Conclude that

$$\langle \bar{\nabla}_{\bar{X}} \bar{Y}, T \rangle = \frac{1}{2} \langle [\bar{X}, \bar{Y}], T \rangle$$

and therefore deduce $\bar{\nabla}_{\bar{X}} \bar{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\bar{X}, \bar{Y}]^V$. where the superscript V denotes the vertical component of a vector field.

(iv): Deduce that if $\pi : M \to N$ is a Riemannian submersion, and $\gamma : [0, 1] \to N$ is a smooth curve and $\bar{\gamma} : [0, 1] \to M$ is a horizontal lift (i.e. $\bar{\gamma}'$ is horizontal and satisfies $\pi \circ \bar{\gamma} = \gamma$) then γ is a geodesic if and only if $\bar{\gamma}$ is. (Bonus question: give a completely different proof of this fact).

Problem 3. S^1 is the set of complex numbers of norm 1; it acts on the round S^{2n+1} in $\mathbb{C}^{n+1}$ by diagonal multiplication on the coordinates. The quotient is $\mathbb{CP}^n$.

The *Fubini-Study* metric on $\mathbb{CP}^n$ is the Riemannian metric for which the quotient map $\pi : S^{2n+1} \to \mathbb{CP}^n$ is a Riemannian submersion.

(i): Check that there is a metric on $\mathbb{CP}^n$ with this property. (Bonus question: write it down explicitly).

(ii): Pick a point p in $\mathbb{CP}^n$ and let γ_1, γ_2 be two unit-speed geodesics with $\gamma_1(0) = \gamma_2(0) = p$ and $\gamma_1'(0) \neq \gamma_2'(0)$. Let T be the least strictly positive number such that $\gamma_1(T) = \gamma_2(T)$. What is T? Does it depend on the choices involved?

Department of Mathematics, University of Chicago, Chicago, Illinois, 60637
E-mail address: dannyc@math.uchicago.edu