RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 5

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 10th.

**Problem 1.** Suppose $M$ is compact, orientable, even dimensional and satisfies $K \geq C > 0$ for some constant $C$, where $K$ is the sectional curvature.

(i): Let $\gamma$ be a closed geodesic in $M$, and let $\nu$ be the normal bundle of $\gamma$. Fix $p \in \gamma$ and let $P : \nu(p) \rightarrow \nu(p)$ be the result of parallel transport around $\gamma$. Show that $P$ fixes a nonzero vector $v_0$.

(ii): Let $V$ be the parallel vector field along $\gamma$ with $V(p) = v$. Show (by using the second variation formula or otherwise) that if $\gamma_s$ is a smooth variation of $\gamma$ with $\partial_s \gamma_s = V$ then $\frac{d}{ds}\lambda_g(\gamma_s)|_{s=0} < 0$. Deduce that $\gamma$ is not a local minimum for length in its free homotopy class.

(iii): Show (e.g. by using the Arzela-Ascoli theorem) that if $M$ is a compact manifold (with no assumptions on the curvature), every nontrivial conjugacy class in $\pi_1(M)$ contains a distance-minimizing geodesic. Deduce *Synge’s Theorem*, which says that a compact, orientable, even dimensional manifold with $K \geq C > 0$ is simply-connected.

**Problem 2.** Let $M$ be a compact Riemannian manifold. Recall that a vector field $X$ on $M$ determines a 1-parameter family of diffeomorphisms $\phi_t : M \rightarrow M$, which moves points along the integral curves of $X$. A vector field $X$ is an *infinitesimal isometry* if the $\phi_t$ are isometries.

For any vector field $X$ on a Riemannian manifold $M$ we can define a derivation $A_X$ by the formula $A_X := \mathcal{L}_X - \nabla_X$ where $\mathcal{L}$ denotes Lie derivative.

(i): Show that $X$ is an infinitesimal isometry if and only if $A_X$ is skew-symmetric; i.e.

$$\langle A_X Y, Z \rangle + \langle Y, A_X Z \rangle = 0$$

for all vector fields $Y, Z \in \mathfrak{X}(M)$. Show that this is equivalent to the condition $\mathcal{L}_X g = 0$ where $g$ is the symmetric 2-tensor defining the metric.

(ii): Let $M$ be a compact manifold with strictly negative sectional curvature everywhere. Show that $M$ admits no nontrivial infinitesimal isometries.

**Problem 3 (Challenging).** Throughout this problem assume that $M$ is a compact, connected manifold.

(i): Let $h$ be a smooth symmetric 2-form (i.e. a section of $S^2 T^* M$). Let $g$ be a Riemannian metric on $M$ (so that $g$ is a smooth symmetric 2-form which is positive definite everywhere). Show that $g + h \theta$ defines a Riemannian metric $g_t$ for all sufficiently small $t$.

(ii): Let $\text{vol}_{g_t}(M)$ denote the volume of $M$ with respect to the $g_t$ metric for $g_t = g + \theta$ as above. Show that $\frac{d}{dt}\text{vol}_{g_t}(M)|_{t=0} = 0$ if and only if $\int_M \text{tr}(h) d\text{vol}_g = 0$.

(iii): Recall that the scalar curvature $s$ is the trace of the Ricci curvature of a Riemannian manifold. If we want to emphasize how $s$ depends on the metric $g$ we write $s_g$. The *total scalar curvature* of the metric $g$, denoted $\mathbb{S}(g)$, is the integral

$$\mathbb{S}(g) := \int_M s_g d\text{vol}_g$$

Show that

$$\frac{d}{dt}\mathbb{S}(g + \theta)|_{t=0} = \int_M \langle s_g/2 \rangle g - \text{Ric}_g, h \rangle_g d\text{vol}_g$$

where $\langle \cdot, \cdot \rangle_g$ denotes the inner product on $S^2 T^*_p M$ for each $p$ induced by the Riemannian metric $g$.

(iv): (Hilbert) Suppose that $M$ is of dimension at least 3. Let $\mathcal{M}_1$ denote the space of smooth metrics $g$ on $M$ for which $\text{vol}_g(M) = 1$. Deduce that $(M, g)$ is a critical point for $\mathbb{S}(\cdot)$ in $\mathcal{M}_1$ if and only if it is Einstein (i.e. if and only if $\text{Ric}_g = \lambda g$ for some constant $\lambda$).