RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 5

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 10th.

Problem 1. Suppose M is compact, orientable, even dimensional and satisfies $K \ge C > 0$ for some constant C, where K is the sectional curvature.
(i): Let γ be a closed geodesic in M, and let ν be the normal bundle of γ. Fix $p \in \gamma$ and let $P : \nu(p) \to \nu(p)$ be the result of parallel transport around γ. Show that P fixes a nonzero vector v.
(ii): Let V be the parallel vector field along γ with $V(p) = v$. Show (by using the second variation formula or otherwise) that if γ_s is a smooth variation of γ with $\partial_s \gamma_s = V$ then $\frac{d}{ds^2}\text{length}(\gamma_s)|_{s=0} < 0$. Deduce that γ is not a local minimum for length in its free homotopy class.
(iii): Show (e.g. by using the Arzela-Ascoli theorem) that if M is a compact manifold (with no assumptions on the curvature), every nontrivial conjugacy class in $\pi_1(M)$ contains a distance-minimizing geodesic. Deduce *Synge's Theorem*, which says that a compact, orientable, even dimensional manifold with $K \ge C > 0$ is simply-connected.

Problem 2. Let M be a compact Riemannian manifold. Recall that a vector field X on M determines a 1-parameter family of diffeomorphisms $\phi_t : M \to M$, which moves points along the integral curves of X. A vector field X is an *infinitesimal isometry* if the ϕ_t are isometries.

For any vector field X on a Riemannian manifold M we can define a derivation A_X by the formula $A_X := \mathcal{L}_X - \nabla_X$ where $\mathcal{L}$ denotes Lie derivative.
(i): Show that X is an infinitesimal isometry if and only if A_X is skew-symmetric; i.e.

$$\langle A_X Y, Z\rangle + \langle Y, A_X Z\rangle = 0$$

for all vector fields $Y, Z \in \mathfrak{X}(M)$. Show that this is equivalent to the condition $\mathcal{L}_X g = 0$ where g is the symmetric 2-tensor defining the metric.
(ii): Let M be a compact manifold with strictly negative sectional curvature everywhere. Show that M admits no nontrivial infinitesimal isometries.

Problem 3 (Challenging). Throughout this problem assume that M is a compact, connected manifold.
(i): Let h be a smooth symmetric 2-form (i.e. a section of S^2T^*M). Let g be a Riemannian metric on M (so that g is a smooth symmetric 2-form which is positive definite everywhere). Show that $g + th$ defines a Riemannian metric g_t for all sufficiently small t.
(ii): Let $\text{vol}_{g_t}(M)$ denote the volume of M with respect to the g_t metric for $g_t = g + th$ as above. Show that $\frac{d}{dt}\text{vol}_{g_t}(M)|_{t=0} = 0$ if and only if $\int_M \text{tr}(h) d\text{vol}_g = 0$.
(iii): Recall that the scalar curvature s is the trace of the Ricci curvature of a Riemannian manifold. If we want to emphasize how s depends on the metric g we write s_g. The *total scalar curvature* of the metric g, denoted $\mathbb{S}(g)$, is the integral

$$\mathbb{S}(g) := \int_M s_g d\text{vol}_g$$

Show that

$$\frac{d}{dt}\mathbb{S}(g + th)|_{t=0} = \int_M \langle (s_g/2)g - \text{Ric}_g, h\rangle_g d\text{vol}_g$$

where $\langle \cdot, \cdot \rangle_g$ denotes the inner product on $S^2T_p^*M$ for each p induced by the Riemannian metric g.
(iv): (Hilbert) Suppose that M is of dimension at least 3. Let $\mathcal{M}_1$ denote the space of smooth metrics g on M for which $\text{vol}_g(M) = 1$. Deduce that (M, g) is a critical point for $\mathbb{S}(\cdot)$ in $\mathcal{M}_1$ if and only if it is Einstein (i.e. if and only if $\text{Ric}_g = \lambda g$ for some constant λ).

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