RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 4

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 3rd.

Problem 1. Let M be a Riemannian manifold, and suppose that the sectional curvature K is *constant* (i.e. it takes the same value on every 2-plane through every point). Show that there is a formula

$$\langle R(X,Y)Z,W\rangle = -K\cdot(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle)$$

Deduce that (under the assumption that M has constant sectional curvature K), if $\gamma(t)$ is a geodesic and $e_i(t)$ are parallel orthonormal vector fields along γ giving a basis for the normal bundle $\nu|_\gamma$, every Jacobi field V along γ with $\langle V(0),\gamma'(0)\rangle = 0$ and $\langle V'(0),\gamma'(0)\rangle = 0$ can be written uniquely in the form

- $V(t) = \sum_i (a_i \sin(t\sqrt{K}) + b_i\cos(t\sqrt{K}))e_i(t)$ if $K>0$;
- $V(t) = \sum_i (a_i t + b_i)e_i(t)$ if $K=0$; and
- $V(t) = \sum_i (a_i \sinh(t\sqrt{-K}) + b_i\cosh(t\sqrt{-K}))e_i(t)$ if $K<0$

for suitable constants a_i, b_i.

Problem 2. If we think of S^3 as the unit sphere in $\mathbb{C}^2$ (with its standard Hermitian metric), multiplication of the coordinates by $e^{i\theta}$ exhibits S^3 as a principal S^1 bundle over S^2 (this is usually known as the *Hopf fibration*). Let ξ be the 1-dimensional (real) subbundle of TS^3 tangent to the S^1 fibers, and let $\xi^\perp$ denote the orthogonal complement, so that $TS^3 = \xi\oplus\xi^\perp$. If g denotes the round metric on S^3, define a 1-parameter family of Riemannian metrics g_t by

$$g_t := g|_{\xi^\perp} \oplus t^2 g|_\xi$$

In other words, the length of vectors tangent to ξ are scaled by t (relative to the g metric), while the length of vectors perpendicular to ξ is the same as in the g metric. Compute the sectional curvature as a function of t. How does the sectional curvature behave in the limit as $t\to 0$ or $t\to\infty$?

(Note: a 3-sphere with one of the metrics g_t is sometimes called a *Berger sphere*)

Problem 3. A *surface of revolution* is a smooth surface in $\mathbb{E}^3$ obtained by rotating a smooth curve (called the *generatrix*) in the x–z plane around the z axis. The generatrix, and the other curves on S obtained by rotating it, are called the *meridians*. Let S be a surface of revolution.
(i): (Clairaut's theorem) Let $\gamma(t)$ be a geodesic on S. Show that the angular momentum of $\gamma(t)$ about the z axis is constant; i.e. if r is the distance to the z-axis, and $\theta(t)$ is the angle between $\gamma'(t)$ and the meridian through $\gamma(t)$, then $r\sin(\theta)$ is constant as a function of t.
(ii): For S the torus obtained by rotating the curve $(x-3)^2+z^2=1$ about the z-axis, give an explicit formula for the geodesics.

(Bonus question: if you want to solve an ODE, why is it helpful to find a conserved quantity — i.e. a function of the dependent variables that is constant on each solution?)

Problem 4. Let M be a Riemannian manifold. The *Ricci curvature tensor* Ric is the 2-tensor

$$\mathrm{Ric}(X,Y) = \text{trace of the map } Z\to R(Z,X)Y$$

(i): Show that $\mathrm{Ric}(X,Y)=\sum_i\langle R(e_i,X)Y,e_i\rangle$ where e_i is any orthonormal basis. Deduce that for any unit vector $v\in T_pM$, the number $\frac{1}{n-1}\mathrm{Ric}(v,v)$ is equal to the average of the sectional curvatures in all 2-planes in T_pM containing v (hint: first say what "average" means here).
(ii): Let M be a 3-manifold. Show that the values of Ric at a point determines the full curvature tensor R at that point. Express this as a statement of representation theory.

(iii): The symmetries of the Ricci curvature tensor imply that it is a symmetric bilinear form on T_pM for each p, just as the Riemannian metric tensor g is. Suppose that M is connected, and that there is some smooth function $f(t)$ so that $\mathrm{Ric} = f(t)g$, so that the Ricci tensor is proportional to the metric tensor at each point, where *a priori* the constant of proportionality can vary from point to point. If the dimension of M is at least 3, show that f is *constant*, so that the constant of proportionality is actually constant. What happens in dimension 2?

A Riemannian manifold satisfying $\mathrm{Ric} = \lambda g$ for some constant λ is said to be *Einstein*. Deduce that for a 3-manifold, the condition of being Einstein is equivalent to having constant sectional curvature.

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