Problem 1. Let $M$ be a Riemannian manifold, and suppose that the sectional curvature $K$ is constant (i.e. it takes the same value on every 2-plane through every point). Show that there is a formula

$$\langle R(X,Y)Z,W \rangle = -K \cdot (\langle X,Z \rangle \langle Y,W \rangle - \langle Y,Z \rangle \langle X,W \rangle)$$

Deduce that (under the assumption that $M$ has constant sectional curvature $K$), if $\gamma(t)$ is a geodesic and $e_i(t)$ are parallel orthonormal vector fields along $\gamma$ giving a basis for the normal bundle $\nu_{\gamma(t)}$, every Jacobi field $V$ along $\gamma$ with $(V(0),\gamma'(0)) = 0$ and $(V'(0),\gamma'(0)) = 0$ can be written uniquely in the form

- $V(t) = \sum (a_i \sin(t\sqrt{K}) + b_i \cos(t\sqrt{K}))e_i(t)$ if $K > 0$;
- $V(t) = \sum (a_i t + b_i)e_i(t)$ if $K = 0$; and
- $V(t) = \sum (a_i \sinh(t\sqrt{-K}) + b_i \cosh(t\sqrt{-K}))e_i(t)$ if $K < 0$

for suitable constants $a_i, b_i$.

Problem 2. If we think of $S^3$ as the unit sphere in $\mathbb{C}^2$ (with its standard Hermitian metric), multiplication of the coordinates by $e^{i\theta}$ exhibits $S^3$ as a principal $S^1$ bundle over $S^2$ (this is usually known as the Hopf fibration). Let $\xi$ be the 1-dimensional (real) subbundle of $TS^3$ tangent to the $S^1$ fibers, and let $\xi^\perp$ denote the orthogonal complement, so that $TS^3 = \xi \oplus \xi^\perp$. If $g$ denotes the round metric on $S^3$, define a 1-parameter family of Riemannian metrics $g_t$ by

$$g_t := g|_{\xi^\perp} \oplus t^2 g|_{\xi}$$

In other words, the length of vectors tangent to $\xi$ are scaled by $t$ (relative to the $g$ metric), while the length of vectors perpendicular to $\xi$ is the same as in the $g$ metric. Compute the sectional curvature as a function of $t$. How does the sectional curvature behave in the limit as $t \to 0$ or $t \to \infty$?

(Note: a 3-sphere with one of the metrics $g_t$ is sometimes called a Berger sphere)

Problem 3. A surface of revolution is a smooth surface in $\mathbb{E}^3$ obtained by rotating a smooth curve (called the generatrix) in the $x$-$z$ plane around the $z$ axis. The generatrix, and the other curves on $S$ obtained by rotating it, are called the meridians. Let $S$ be a surface of revolution.

(i): (Clairaut’s theorem) Show that if $r$ is the distance to the $z$-axis, and $\theta(t)$ is the angle between $\gamma'(t)$ and the meridian through $\gamma(t)$, then $r \sin(\theta)$ is constant as a function of $t$.

(ii): For $S$ the torus obtained by rotating the curve $(x - 3)^2 + z^2 = 1$ about the $z$-axis, give an explicit formula for the geodesics.

(Bonus question: if you want to solve an ODE, why is it helpful to find a conserved quantity — i.e. a function of the dependent variables that is constant on each solution?)

Problem 4. Let $M$ be a Riemannian manifold. The Ricci curvature tensor $\text{Ric}$ is the 2-tensor

$$\text{Ric}(X,Y) = \text{trace of the map } Z \to R(Z,X)Y$$

(i): Show that $\text{Ric}(X,Y) = \sum_i (R(e_i,X)Y) e_i$ where $e_i$ is any orthonormal basis. Deduce that for any unit vector $v \in T_pM$, the number $\frac{1}{n-1} \text{Ric}(v,v)$ is equal to the average of the sectional curvatures in all 2-planes in $T_pM$ containing $v$ (hint: first say what “average” means here).

(ii): Let $M$ be a 3-manifold. Show that the values of $\text{Ric}$ at a point determines the full curvature tensor $R$ at that point. Express this as a statement of representation theory.
The symmetries of the Ricci curvature tensor imply that it is a symmetric bilinear form on $T_p M$ for each $p$, just as the Riemannian metric tensor $g$ is. Suppose that $M$ is connected, and that there is some smooth function $f(t)$ so that $\text{Ric} = f(t)g$, so that the Ricci tensor is proportional to the metric tensor at each point, where a priori the constant of proportionality can vary from point to point. If the dimension of $M$ is at least 3, show that $f$ is constant, so that the constant of proportionality is actually constant. What happens in dimension 2?

A Riemannian manifold satisfying $\text{Ric} = \lambda g$ for some constant $\lambda$ is said to be Einstein. Deduce that for a 3-manifold, the condition of being Einstein is equivalent to having constant sectional curvature.