RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 3

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due April 26th.

Problem 1. Give an example of a Riemannian metric on $\mathbb{R}^2$ which is complete but has *finite* total area.

Problem 2. Suppose s_i are local sections of a smooth bundle E, and ∇ is a connection on E for which we can write (in terms of these coordinates) $\nabla = d + \omega$ where ω is a matrix of 1-forms (with components ω_{ij}). Express R in the same coordinates as a matrix of 2-forms Ω, and show that

$$\Omega = d\omega - \omega \wedge \omega$$

How does Ω transform if we change coordinates on E locally to $s_i' := \sum g_{ij} s_j$? What does this have to do with R being a tensor?

Problem 3. (i): Let G be a group of (real or complex) $n \times n$ matrices, thought of as a subspace of $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$ with coordinates given by the entries. In each of the following cases, show that G is a smooth submanifold of $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$, and determine the tangent space at the identity as a vector space of the space of $n \times n$ (real or complex) matrices (this tangent space at the identity matrix is denoted $\mathfrak{g}$, and called the *Lie algebra* of the *Lie group* G).

- $G = \mathrm{GL}(n)$, the group of invertible $n \times n$ matrices.
- $G = \mathrm{SL}(n)$, the group of invertible $n \times n$ matrices with determinant 1.
- $G = \mathrm{O}(n)$, the group of invertible $n \times n$ matrices satisfying $A^T = A^{-1}$.
- $G = \mathrm{Sp}(2n)$, the group of invertible $2n \times 2n$ matrices satisfying $A^T J A = J$ where $J := \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.
- $G = \mathrm{U}(n)$, the group of invertible $n \times n$ *complex* matrices satisfying $A^* = A^{-1}$ (where A^* denotes the complex conjugate of the transpose).

(ii): Let E be a smooth (real or complex) bundle over M with a G-structure where G is one of the groups above. This means that E admits a collection of local trivializations where the transition functions between two trivializations on each fiber are contained in G. Say that a connection ∇ is *compatible* with the G structure if parallel transport induces an automorphism of fibers represented by an element of G (with respect to one of the local trivializations). Show that this is equivalent to the condition that, in any of the local trivializations, ∇ can be expressed in the form $\nabla = d + \omega$ where ω is a 1-form with coefficients in $\mathfrak{g}$.

(iii): Let E be a smooth bundle over M with a G-structure, and suppose ∇ is compatible with the G structure. Show that R can be expressed in local coordinates as a matrix Ω of 2-forms with coefficients in $\mathfrak{g}$. Now suppose that $P : \mathfrak{g} \to \mathbb{C}$ is a homogeneous polynomial of degree m in the entries of $\mathfrak{g}$ which is invariant under conjugation by G. Deduce that $P(\Omega)$ is a well-defined $2m$-form on M, independent of the choice of local trivialization.

Problem 4. Show directly that the Riemann curvature tensor (for the Levi-Civita connection on TM) can be recovered from the values of the sectional curvature, by giving an explicit formula for $\langle R(X,Y)Z, W\rangle$ in terms of K.

Problem 5. Let C be the circle in the x–z plane defined by the equation $(x-3)^2 + z^2 = 1$, and let T be the surface in $\mathbb{E}^3$ obtained by revolving C around the z-axis. For each point on T give a formula for the size and the directions of the principal curvatures, and the sectional curvature. What is the integral of the sectional curvature over T?

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