RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 3

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due April 26th.

Problem 1. Give an example of a Riemannian metric on $\mathbb{R}^2$ which is complete but has finite total area.

Problem 2. Suppose $s_i$ are local sections of a smooth bundle $E$, and $\nabla$ is a connection on $E$ for which we can write (in terms of these coordinates) $\nabla = d + \omega$ where $\omega$ is a matrix of 1-forms (with components $\omega_{ij}$). Express $R$ in the same coordinates as a matrix of 2-forms $\Omega$, and show that

$$\Omega = d\omega - \omega \wedge \omega$$

How does $\Omega$ transform if we change coordinates on $E$ locally to $s'_i := \sum g_{ij} s_j$? What does this have to do with $R$ being a tensor?

Problem 3. (i): Let $G$ be a group of (real or complex) $n \times n$ matrices, thought of as a subspace of $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$ with coordinates given by the entries. In each of the following cases, show that $G$ is a smooth submanifold of $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$, and determine the tangent space at the identity as a vector space of the space of $n \times n$ (real or complex) matrices (this tangent space at the identity matrix is denoted $\mathfrak{g}$, and called the Lie algebra of the Lie group $G$).

- $G = \text{GL}(n)$, the group of invertible $n \times n$ matrices.
- $G = \text{SL}(n)$, the group of invertible $n \times n$ matrices with determinant 1.
- $G = \text{O}(n)$, the group of invertible $n \times n$ matrices satisfying $A^T = A^{-1}$.
- $G = \text{Sp}(2n)$, the group of invertible $2n \times 2n$ matrices satisfying $A^T J A = J$ where $J := \begin{vmatrix} 0 & I \\ -I & 0 \end{vmatrix}$.
- $G = \text{U}(n)$, the group of invertible $n \times n$ complex matrices satisfying $A^* = A^{-1}$ (where $A^*$ denotes the complex conjugate of the transpose).

(ii): Let $E$ be a smooth (real or complex) bundle over $M$ with a $G$-structure where $G$ is one of the groups above. This means that $E$ admits a collection of local trivializations where the transition functions between two trivializations on each fiber are contained in $G$. Say that a connection $\nabla$ is compatible with the $G$ structure if parallel transport induces an automorphism of fibers represented by an element of $G$ (with respect to one of the local trivializations). Show that this is equivalent to the condition that, in any of the local trivializations, $\nabla$ can be expressed in the form $\nabla = d + \omega$ where $\omega$ is a 1-form with coefficients in $\mathfrak{g}$.

(iii): Let $E$ be a smooth bundle over $M$ with a $G$-structure, and suppose $\nabla$ is compatible with the $G$ structure. Show that $R$ can be expressed in local coordinates as a matrix $\Omega$ of 2-forms with coefficients in $\mathfrak{g}$. Now suppose that $P : \mathfrak{g} \to \mathbb{C}$ is a homogeneous polynomial of degree $m$ in the entries of $\mathfrak{g}$ which is invariant under conjugation by $G$. Deduce that $P(\Omega)$ is a well-defined $2m$-form on $M$, independent of the choice of local trivialization.

Problem 4. Show directly that the Riemann curvature tensor (for the Levi-Civita connection on $TM$) can be recovered from the values of the sectional curvature, by giving an explicit formula for $\langle R(X, Y)Z, W \rangle$ in terms of $K$.

Problem 5. Let $C$ be the circle in the $x$–$z$ plane defined by the equation $(x - 3)^2 + z^2 = 1$, and let $T$ be the surface in $\mathbb{E}^3$ obtained by revolving $C$ around the $z$-axis. For each point on $T$ give a formula for the size and the directions of the principal curvatures, and the sectional curvature. What is the integral of the sectional curvature over $T$?
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