Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due April 19th.

**Problem 1.** Compute the first variation of energy of a smooth 1-parameter variation of a smooth curve \( \gamma : [0, 1] \to M \) in a Riemannian manifold \( M \). Deduce that (among all smooth curves with fixed endpoints) the critical points of the energy functional are the geodesics.

**Problem 2.** Let \( C \) be the cone \( x^2 + y^2 = z^2 \) in Euclidean \( \mathbb{E}^3 \), which is smooth away from the point \((0, 0, 0)\). Determine the geodesics on this cone (as smooth curves in \( \mathbb{E}^3 \)) by directly solving the geodesic equations. Now slit the cone open along the ray \( x = z \) and lay it flat in the plane (by “unrolling” it); what do the geodesics look like when the cone is laid flat in the plane?

**Problem 3.** If \( \Sigma \) is a smooth (2-dimensional) oriented surface in \( \mathbb{E}^3 \), the Gauss map is a map \( g : \Sigma \to S^2 \) (the unit sphere in \( \mathbb{E}^3 \)) defined uniquely by the property that \( T_p \Sigma \) and \( T_{g(p)} S^2 \) are parallel (in \( \mathbb{E}^3 \)) as oriented planes. Show that \( T \Sigma \) can be naturally identified with the pullback \( g^* T S^2 \). Derive the following consequence: the pullback of the Gauss map commutes with parallel transport; i.e. if \( \gamma : [0, 1] \to \Sigma \) is a smooth curve, and \( V \in \Gamma(TS^2) \) is parallel along \( g \circ \gamma \) (i.e. \( \nabla_{(g \circ \gamma)'} (V) = 0 \)) then the pullback \( g^* V \) (as a section of \( T \Sigma \)) is parallel along \( \gamma \).

**Problem 4.** If the Riemannian metric is expressed locally in coordinates \( x_i \) in the form \( g := \sum g_{ij} dx_i dx_j \), derive a formula for the Christoffel symbols \( \Gamma^k_{ij} \) (in the same coordinates) in terms of the \( g_{ij} \).

**Problem 5.** Let \( x_i \) be geodesic normal coordinates centered at a point \( p \) (i.e. obtained from the exponential map by exponentiating orthonormal linear coordinates on \( T_p M \)). Show that the metric \( g_{ij} dx_i dx_j \) in these coordinates satisfies

\[
g_{ij}(p) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]

and \( (\partial_k g_{ij})(p) = 0 \) for all \( i, j, k \). In other words, the metric “oscillates” the Euclidean metric to first order (at \( p \)).