RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 2

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due April 19th.

Problem 1. Compute the first variation of energy of a smooth 1-parameter variation of a smooth curve $\gamma : [0,1] \to M$ in a Riemannian manifold M. Deduce that (among all smooth curves with fixed endpoints) the critical points of the energy functional are the geodesics.

Problem 2. Let C be the cone $x^2+y^2=z^2$ in Euclidean $\mathbb{E}^3$, which is smooth away from the point $(0,0,0)$. Determine the geodesics on this cone (as smooth curves in $\mathbb{E}^3$) by directly solving the geodesic equations. Now slit the cone open along the ray $x=z$ and lay it flat in the plane (by "unrolling" it); what do the geodesics look like when the cone is laid flat in the plane?

Problem 3. If Σ is a smooth (2-dimensional) oriented surface in $\mathbb{E}^3$, the *Gauss map* is a map $g : \Sigma \to S^2$ (the unit sphere in $\mathbb{E}^3$) defined uniquely by the property that $T_p\Sigma$ and $T_{g(p)}S^2$ are parallel (in $\mathbb{E}^3$) as *oriented* planes. Show that $T\Sigma$ can be naturally identified with the pullback g^*TS^2. Derive the following consequence: the pullback of the Gauss map commutes with parallel transport; i.e. if $\gamma : [0,1] \to \Sigma$ is a smooth curve, and $V \in \Gamma(TS^2)$ is parallel along $g \circ \gamma$ (i.e. $\nabla_{(g\circ\gamma)'}(V) = 0$) then the pullback g^*V (as a section of $T\Sigma$) is parallel along γ.

Problem 4. If the Riemannian metric is expressed locally in coordinates x_i in the form $g := \sum g_{ij}dx_idx_j$, derive a formula for the Christoffel symbols Γ_{ij}^k (in the same coordinates) in terms of the g_{ij}.

Problem 5. Let x_i be geodesic normal coordinates centered at a point p (i.e. obtained from the exponential map by exponentiating orthonormal linear coordinates on T_pM). Show that the metric $g_{ij}dx_idx_j$ in these coordinates satisfies

$$g_{ij}(p) = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ otherwise} \end{cases}$$

and $(\partial_k g_{ij})(p) = 0$ for all i,j,k. In other words, the metric "osculates" the Euclidean metric to first order (at p).

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