# RIEMANNIAN GEOMETRY, SPRING 2013, HOMEWORK 1

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due April 12th.

*Problem* 1. Let $M$ be an $n$-dimensional Riemannian manifold. Show that for every point $p$ there is an open neighborhood $U$ around $p$ and a diffeomorphism $f$ from $U$ to an open subset of $\mathbb{E}^n$ which is *short*; i.e. such that for all vectors $u \in T_qM$ where $q \in U$, we have

$$\langle u, u\rangle_q \ge \langle df(u), df(u)\rangle_{f(q)}$$

where the right hand side is the usual Euclidean inner product. Deduce that the Riemannian metric on $M$ makes it into a genuine *metric space*; i.e. that for any two distinct points $r, s \in M$ the infimum of the length of paths joining $r$ to $s$ is positive.

*Problem* 2. If $\nabla$ is a connection on a smooth bundle $E$, covariant differentiation $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is tensorial in the first term, but not in the second. However, if $\nabla_1$, $\nabla_2$ are two connections on $E$, show that the difference $\nabla_1 - \nabla_2$ *is* a tensor in the second term. Thus, if we choose a local trivialization of $E$, deduce that any covariant derivative on $E$ can be expressed locally (in terms of this trivialization) as $d + \omega$, where $\omega$ is a matrix of 1-forms. I.e. if $s_i$ are local sections of $E$, we can write

$$\nabla\left(\sum_i f_i s_i\right) = \sum_i df_i \otimes s_i + \sum_{i,j} f_i \omega_{ij} \otimes s_j$$

*Problem* 3. For the round unit sphere in $\mathbb{E}^3$ with its induced Riemannian metric, express the Levi-Civita connection on the sphere in terms of local polar coordinates.

*Problem* 4. Check that the definition of the Levi-Civita connection given (implicitly) by the formula

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle$$

satisfies the properties of a connection.

*Problem* 5. Let $\nabla$ be a connection on $TM$, which thereby induces a connection on $T^*M$ in the usual way. Show that $\nabla$ on $TM$ is torsion-free if and only if the composition

$$\Gamma(T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M) \xrightarrow{\pi} \Gamma(\Lambda^2 T^*M)$$

is equal to exterior $d$, where $\Gamma$ denotes the space of smooth sections, and $\pi$ is the antisymmetrizing map.

*Problem* 6. Show that a connection $\nabla$ on $TM$ preserves the metric if and only if the metric 2-tensor $g \in \Gamma(S^2T^*M)$ is parallel; i.e. $\nabla g = 0$.

*Problem* 7. Give an example of a connection on Euclidean $\mathbb{E}^3$ which preserves the metric, but is not torsion-free.

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