RIEMANNIAN GEOMETRY, SPRING 2013, FINAL EXAM

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This final exam was posted online on Friday, May 31st. Answers should be handed in to Bena Tshishiku by 1:30pm on Friday, June 7th. You are *not* allowed to collaborate with other students, or to use resources found on the internet or elsewhere, to do these problems, except for the online course notes on the course webpage, and your own notes.

Problem 1. (i): Let M be a Riemannian manifold, and let $\alpha \in \Omega^p(M)$ be *parallel*; i.e. suppose $\nabla\alpha = 0$ where ∇ is the Levi-Civita connection. Show that α is *closed*; i.e. $d\alpha = 0$.
(ii): Let M be a Riemannian manifold. If we pick a basepoint $p \in M$, then for any smooth path $\gamma : [0,1] \to M$, parallel transport along γ induces an automorphism $g_\gamma \in \text{Aut}(T_pM) \cong \text{O}(2n,\mathbb{R})$. The *holonomy group* G_p is the subgroup of $\text{Aut}(T_pM)$ generated by all such loops γ. One definition of a *Kähler manifold* is a $2n$-dimensional Riemannian manifold M such that the holonomy group G_p is conjugate into $\text{U}(n) \subset \text{O}(2n,\mathbb{R})$. Show that a Kähler manifold is *symplectic* — i.e. that there is a closed 2-form ω such that ω^n is nowhere vanishing.

Problem 2. Show that there is no Riemannian metric on the 2-ball B^2 which has negative sectional curvature everywhere, and such that the boundary is *concave*; i.e. the geodesic curvature along the boundary points outwards (this means that for any smooth curve γ in the boundary, the part of $\nabla_{\gamma'}\gamma'$ normal to the boundary points outwards).

(Bonus problem: (Hass) show that there *is* a Riemannian metric on the 3-ball B^3 which has negative sectional curvature everywhere, and such that the boundary is concave.)

Problem 3. Recall that the *mean curvature* of a surface in $\mathbb{R}^3$ is the trace of the second fundamental form. The *Scherk surface* in $\mathbb{R}^3$ is defined by the equation $e^z\cos(x) - \cos(y) = 0$.
(i): Show that the mean curvature of the Scherk surface vanishes everywhere. (A surface with this property is said to be a *minimal surface*, and is a critical point for area for compactly supported variations).
(ii): The coordinate functions x, y, z on $\mathbb{R}^3$ restrict to smooth functions on the Scherk surface. Show that they are harmonic.

(Bonus problem: what is the topology of the Scherk surface? I.e. describe it as a surface of some genus with some number of punctures or boundary components.)

Problem 4. (Bott) Let G be a compact Lie group with a bi-invariant metric. Let p be a point, and let q be conjugate to p along a geodesic γ. Show that the dimension of the space of Jacobi fields along γ vanishing at p and q is even.

Problem 5. (Chevalley-Eilenberg) Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. For each p, let Ω^p_G denote the subset of p-forms on G which are left-invariant.
(Bonus problem (i): Show that (Ω^p_G, d) is a complex, and its homology is isomorphic to the de Rham cohomology (i.e. the homology of the complex of all smooth forms (Ω^p, d)). (Hint: define a retraction $\Omega^p \to \Omega^p_G$ by averaging over G).)
(ii): Using Cartan's formula, show that for α in Ω^p_G, we can obtain a formula for $d\alpha$ using just the Lie algebra structure on $\mathfrak{g}$. (Hint: evaluate α and $d\alpha$ on left-invariant vector fields).
(iii): Deduce that the (real) cohomology of G depends only on the algebraic structure of $\mathfrak{g}$. Hence, if G, G' are compact, connected Lie groups with isomorphic Lie algebras, then $H^*(G;\mathbb{R}) \cong H^*(G';\mathbb{R})$.
(iv): Give two nontrivial examples of pairs of compact, connected Lie groups with isomorphic Lie algebras, and show that their (real) cohomology groups are isomorphic, to check (iii).

Problem 6. Let S be a Riemannian surface (i.e. a 2-dimensional manifold) with metric g. Define a new metric $\tilde{g}$ by $\tilde{g} = e^f g$ for some smooth function f. If $s_{\tilde{g}}$ and s_g are the scalar curvatures of the two metrics (thought of as functions on S) show that

$$s_{\tilde{g}} = e^{-f}(\Delta_g f + s_g)$$

where Δ_g denotes the Laplacian (on functions) in the g-metric.

Problem 7. (Weinberger) Let G be a finite group of diffeomorphisms of the torus T^n fixing some point p (i.e. p is fixed by every element of G). Show that the action of G on $H_1(T^n;\mathbb{Z}) = \mathbb{Z}^n$ is faithful. (Hint: show that G preserves some Riemannian metric on T^n. Use the Hodge theorem, and integration of (harmonic) 1-forms along paths from some base point to get a (degree 1) map from T^n to $H_1(T^n;\mathbb{R})/H_1(T^n;\mathbb{Z}) = T^n$ that commutes with a suitable action of G.) (Bonus problem: what if G is not assumed to fix a point p?)

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