LINEAR ALGEBRA, WINTER 2018, MIDTERM

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This midterm exam was given in class on Thursday, February 7. When a problem asks you to give an example of something you must **also explain why the example you give satisfies the demands of the problem**.

Problem 1. Let M be an $n \times n$ matrix with real entries. The matrix M determines a linear map $L_{\mathbb{R}}(M)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ in the usual way. Let's suppose $L_{\mathbb{R}}(M)$ is invertible.

Since every real number is also a complex number, we can think of M as an $n \times n$ matrix with complex entries (which just happen to be real). It determines a linear map $L_{\mathbb{C}}(M)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$.

Must $L_{\mathbb{C}}(M)$ be invertible? (hint: what could its inverse be?)

Problem 2. Let V denote the vector space of *real homogeneous degree 1 polynomials in two variables x and y*; i.e. polynomials of the form $ax + by$ where $a, b \in \mathbb{R}$. Let W denote the vector space of *homogeneous degree 3 polynomials in two variables x and y*; i.e. polynomials of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Let $\phi : V \to W$ be the map that takes a polynomial p to its cube; i.e. $\phi(p) = p^3$. Is ϕ linear?

Problem 3. Let $\mathbb{F}_3$ denote the field of 'integers modulo 3'; i.e. it consists of the number $0, 1, 2$ with addition $0 + x = x$, $1 + 1 = 2$, $1 + 2 = 0$, $2 + 2 = 1$ and multiplication $0 * x = 0$, $1 * x = x$, $2 * 2 = 1$.

Just as in problem 1, let V_3 denote the vector space of homogeneous degree 1 polynomials in two variables x and y but *with coefficients in* $\mathbb{F}_3$, and likewise let W_3 denote the vector space of homogeneous degree 3 polynomials in two variables x and y with coefficients in $\mathbb{F}_3$. Let $\phi_3 : V_3 \to W_3$ denote the map that takes a polynomial p to its cube. Is ϕ_3 linear?

Problem 4. Let Γ be the directed graph in Figure 1.

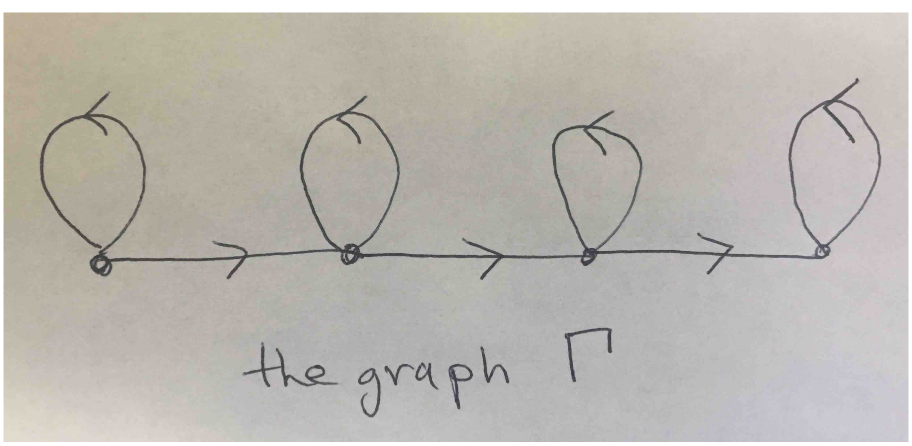

FIGURE 1. The directed graph Γ

(1) write down an adjacency matrix M for Γ
(2) write down M^{1000}.
(3) how many directed paths are there in Γ of length 1000?

Problem 5. Consider the following system of linear equations in the variables x_1, x_2, x_3, x_4, x_5 (Note: none of the equations involves x_5!)

$$\begin{aligned} 7x_1 - x_2 + 7x_4 &= 2 \\ x_1 + 5x_3 &= 7 \\ 9x_1 + 9x_2 + 9x_3 + 9x_4 &= 9 \end{aligned}$$

(1) Without solving the equations, say if it's possible there is a unique solution.
(2) Without necessarily solving the equations, say if there's at least one solution.
(3) Find all solutions!

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