

CHARACTERISTIC CLASSES, WINTER 2016, MIDTERM

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This midterm exam was posted online on Friday, February 5, and is due before class Friday, February 12. Collaboration is not allowed, nor is the use of outside materials and textbooks. Milnor-Stasheff and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Problem 1. Show by an explicit construction that the tangent bundle of $S^2 \times S^1$ is trivial.

- Problem 2.*
- (1) Let W be a closed, smooth, orientable n -manifold. Show that every class in $H_{n-1}(W; \mathbb{Z})$ (resp. $H_{n-2}(W; \mathbb{Z})$) is represented by a closed, smooth, orientable submanifold of codimension 1 (resp. 2). (Hint: use the fact that S^1 is a $K(\mathbb{Z}, 1)$ (resp. $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$) and represent a cohomology class by a smooth map; then use general position.)
 - (2) Let W be a closed, smooth, orientable $2n$ -manifold, and let M be a closed, smooth, orientable n -submanifold. Let $[M]^\#$ denote the cohomology class (Poincaré) dual to the homology class $[M]$. Geometrically, if M' is another n -submanifold, $[M]^\#([M'])$ counts the number of intersections of M' with M (after perturbing them to be in general position), counted with sign. Deduce that $[M]^\#([M])$ is congruent to $w_n(\nu_M)([M]) \bmod 2$, where ν_M is the normal bundle of the embedding of M in W .
 - (3) Let W be a closed, smooth, simply-connected 4-manifold. Show that w_2 (i.e. the 2nd Stiefel-Whitney class of the tangent bundle) is the unique class in $H^2(W; \mathbb{Z}/2)$ such that $w_2 \cup x = x \cup x$ for all $x \in H^2(W; \mathbb{Z}/2)$.
 - (4) Compute the Stiefel-Whitney classes of the tangent bundle of $\mathbb{C}P^2$.

- Problem 3.*
- (1) Show that every closed 2-manifold can be immersed in \mathbb{R}^3 . Compute the Stiefel-Whitney classes of the tangent bundle in every case and therefore the Stiefel-Whitney numbers of the surfaces. (Hint: recall the classification of closed 2-manifolds, that says every closed connected surface is a sphere, or the connected sum of $g \geq 1$ tori, or the connected sum of $k \geq 1$ projective planes, and that the surface is orientable exactly in the first two cases.)
 - (2) Thinking of $\mathbb{R}P^2$ as the Grassmannian $G_2(3)$ of two-planes in \mathbb{R}^3 , describe its Schubert cell decomposition, and identify w_1 and w_2 of the tautological bundle as explicit cocycles.
 - (3) Draw an explicit immersion of a Klein bottle K in \mathbb{R}^3 . Your immersion determines a map from K to $\mathbb{R}P^2$. Figure out how to “count” the value of $w_1(TK)$ and $w_2(TK)$ on 1- and 2-dimensional cycles in K by how their image intersects Schubert cells in the Grassmannian in general position.

Problem 4. Prove Theorem 10.4 in Milnor-Stasheff, with details (you may consult the argument in the book, and copy it, providing you understand it).

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