DIFFERENTIAL TOPOLOGY, WINTER 2016, HOMEWORK 8

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due March 11th.

Problem 1. Let $f, g : M \to N$ be smoothly homotopic smooth maps. Prove directly (i.e. by constructing a suitable chain homotopy between complexes of forms) that f and g induce the same maps on de Rham cohomology.

Problem 2. Suppose α and β are closed forms on M which have integral periods; i.e. for all $[A] \in H_*(M;\mathbb{Z})$ represented by a smooth cycle A we have $\int_A \alpha \in \mathbb{Z}$, and similarly for β. Show that $\alpha \wedge \beta$ is closed, and has integral periods.

Problem 3. Let $[f] \in \pi_{2n-1}(S^n)$ with $n > 1$. We define the *Hopf invariant* of the class $[f]$ as follows. Choose a smooth representative $f : S^{2n-1} \to S^n$. Let ω be a smooth n-form on S^n with $\int_{S^n} \omega = 1$.

(1) Show that $f^*\omega = d\alpha$ for some $(n-1)$-form α on S^{2n-1}.
(2) With notation as above, define $H([f]) = \int_{S^{2n-1}} \alpha \wedge d\alpha$. Show H is independent of all choices, and defines a map $H : \pi_{2n-1}(S^n) \to \mathbb{R}$.
(3) Show that H, so defined, is a homomorphism.
(4) If S^3 is thought of as the unit sphere in $\mathbb{C}^2$, the *Hopf fibration* is the map $S^3 \to \mathbb{C}P^1 = S^2$ induced by $\mathbb{C}^2 - 0 \to \mathbb{C}P^1$. Compute the Hopf invariant of this element of $\pi_3(S^2)$.
(5) (Bonus problem) show that H takes values in $\mathbb{Z}$.

Problem 4. Construct an explicit smooth homeomorphism from S^2 (the unit sphere in $\mathbb{R}^3$) to the boundary of a regular octahedron.

Problem 5. Let D^2 denote the closed unit disk in $\mathbb{R}^2$. Let $\omega := dx \wedge dy$ denote the standard area form on $\mathbb{R}^2$ (and on D^2 by restriction). Let ϕ be a diffeomorphism of D^2 which is equal to the identity in a neighborhood of ∂D^2, and which preserves area; i.e. $\phi^*\omega = \omega$. We denote the group of such diffeomorphisms by $\text{Diff}_\omega(D^2, \partial D^2)$.

(1) Show that there is a 1-form α with $d\alpha = \omega$. Show that $\phi^*\alpha - \alpha$ is exact, and equal to df for some smooth function f which vanishes on ∂D.
(2) The *Calabi invariant* of ϕ is the integral
$$C(\phi) = \int_{D^2} f\omega$$
with f as above. Show that this does not depend on the choices above. Thus it defines a map $C : \text{Diff}_\omega(D^2, \partial D^2) \to \mathbb{R}$.
(3) Show that C is a homomorphism. Show that it is nontrivial (e.g. by calculation on some example).

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