NOTES ON COMPLEX DYNAMICS

DANNY CALEGARI

Abstract. These are notes on complex dynamics, based on a graduate course taught at the University of Chicago in Winter 2022

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1. Fatou and Julia Sets

1.1. Schwarz Lemma and Montel's Theorem.

1.1.1. *Schwarz Lemma.* The most important theorem in complex analysis is the

Theorem 1.1 (Schwarz Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(0) = 0$. Then $|f'(0)| \le 1$ with equality if and only if f is a rotation.*

Proof. Since f has a zero at 0, the function $g(z) := f(z)/z$ on $\mathbb{D} - 0$ has a removable singularity at 0, and therefore extends to an analytic function on $\mathbb{D}$. By the maximum principle, $|g|$ attains its maximum on $\partial\mathbb{D}$ where it is equal to $|f|$, so $|g| \le 1$ everywhere; i.e. $|f(z)| \le |z|$. The inequality $|f'(0)| \le 1$ follows.

On the other hand, also by the maximum principle, if $|g| = 1$ anywhere in $\mathbb{D}$ then $|g| = 1$ everywhere in $\mathbb{D}$ in which case g is constant of norm 1 (i.e. f is a rotation). Otherwise $|f(z)| < |z|$ on the closed disk of radius $1/2$ (say), so that $|f(z)| < (1-\epsilon)|z|$ on that disk for some positive ϵ, so that $|f'(0)| < 1$. $\square$

1.1.2. *Hyperbolic surfaces.* If f is a holomorphic automorphism of $\mathbb{D}$ fixing 0 then so is f^{-1}; since the product of the derivatives of f and f^{-1} at 0 is 1 it follows that f is a rotation. The group $\mathrm{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$. For example,

$$z \to \frac{z-\alpha}{1-\bar{\alpha}z}$$

is an automorphism taking $\alpha \in \mathbb{D}$ to 0, with inverse $z \to (z+\alpha)/(1+\bar{\alpha}z)$.

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Since the point stabilizers are compact and act transitively on the unit tangent circle at a point, there is a unique $\mathrm{Aut}(\mathbb{D})$-invariant Riemannian metric on $\mathbb{D}$, up to scale — the hyperbolic metric — usually normalized as

$$ds = \frac{2|dz|}{1-|z|^2}$$

We say a Riemann surface X is *hyperbolic* if its universal cover $\tilde{X}$ is uniformized by $\mathbb{D}$. Such an X inherits a unique hyperbolic metric from $\mathbb{D}$. By the uniformization theorem, a surface X is hyperbolic if and only if it is not isomorphic to one of the following:

(1) the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$;
(2) $\mathbb{C}$;
(3) $\mathbb{C}^* := \mathbb{C} - 0$;
(4) a torus $E := \mathbb{C}/\Lambda$ for some $\Lambda := \{m + n\tau : m, n \in \mathbb{Z}\}$, where $\Im(\tau) > 0$.

Surfaces of type (3) and (4) are not simply-connected; their universal cover is isomorphic to $\mathbb{C}$.

Lemma 1.2. *Any holomorphic map $f : X \to Y$ between hyperbolic surfaces is 1-Lipschitz for their respective hyperbolic metrics, and is strictly contracting on compact subsets unless it is a covering map (equivalently: a local isometry).*

Proof. Lift $f : X \to Y$ to $\tilde{f} : \mathbb{D} \to \mathbb{D}$. For every $p \in \mathbb{D}$ let $\alpha, \beta \in \mathrm{Aut}(\mathbb{D})$ be such that $\alpha(0) = p$ and $\beta(\tilde{f}(p)) = 0$. Then $g := \beta\tilde{f}\alpha$ takes 0 to 0 so by the Schwarz Lemma, $|g'(0)| \le 1$ with equality if and only if $g \in \mathrm{Aut}(\mathbb{D})$. Since α and β are isometries in the hyperbolic metric, it follows that $\tilde{f}$ (and therefore also f) is 1-Lipschitz in the hyperbolic metric, and is uniformly strictly contracting on compact subsets unless it is an isometry. □

Lemma 1.3. *Let Y be hyperbolic and X non-hyperbolic Riemann surfaces. Then any holomorphic $f : X \to Y$ is constant.*

Proof. Lift f to $\tilde{f} : \tilde{X} \to \mathbb{D}$ where $\tilde{X}$ is the universal cover of X. Since X is not hyperbolic, $\tilde{X}$ is either $\mathbb{C}$ or $\hat{\mathbb{C}}$, so without loss of generality $\tilde{f}$ restricts to a bounded entire function which is therefore constant. □

1.1.3. *Normal families and Montel's Theorem.*

Definition 1.4 (Normal family)**.** Let $\Omega \subset \hat{\mathbb{C}}$ be a domain (i.e. an open subset). A family $\mathcal{F}$ of holomorphic functions from Ω to $\hat{\mathbb{C}}$ is *normal* in Ω if every sequence $f_n \in \mathcal{F}$ contains a subsequence that converges uniformly on compact subsets of Ω.

A family $\mathcal{F}$ is *normal at a point* z if it is normal in some neighborhood of z.

Lemma 1.5. *A family $\mathcal{F}$ is normal in Ω if and only if it is normal at each point of Ω.*

Proof. One direction is clear. So suppose $\mathcal{F}$ is normal at each point of Ω. Cover Ω by countably many compact subsets D_j. By compactness we may cover each D_j by finitely many U_{ij} so that $\mathcal{F}$ is normal in each U_{ij}. Any sequence $f_n \in \Omega$ has a subsequence that converges uniformly on compact subsets of each U_{ij}, so a further subsequence converges uniformly on D_j. Thus a diagonal subsequence converges uniformly on every compact subset of Ω, so that $\mathcal{F}$ is normal. □

'Montel's Theorem' can refer to one of several related theorems that give sufficient (and, sometimes, necessary) conditions for a family to be normal. We shall use the following formulation:

Theorem 1.6 (Montel's Theorem)**.** *Let $\Omega \subset \hat{\mathbb{C}}$ be a domain, let $x, y, z \in \hat{\mathbb{C}}$ be three distinct values, and let $\mathcal{F}$ be a family of holomorphic functions on Ω taking values in $\hat{\mathbb{C}} - \{x, y, z\}$. Then $\mathcal{F}$ is normal.*

Proof. The Riemann surface $\hat{\mathbb{C}} - \{x, y, z\}$ is hyperbolic. If Ω is non-hyperbolic, then every holomorphic map from Ω to $\hat{\mathbb{C}} - \{x, y, z\}$ is constant, and the theorem is obvious. Otherwise Ω is hyperbolic, so every holomorphic function to $\hat{\mathbb{C}} - \{x, y, z\}$ is 1-Lipschitz in the respective hyperbolic metrics. Thus $\mathcal{F}$ is equicontinuous on Ω so by Arzela–Ascoli $\mathcal{F}$ is normal on Ω and the theorem is proved. □

If U, V are open domains in $\hat{\mathbb{C}}$ and $\mathcal{F}$ is a normal family of holomorphic maps from U to V, then the limit of a subsequence of $\mathcal{F}$ need not map U to V; however it must map U to $\bar{V}$.

1.2. **Fatou and Julia sets.**

1.2.1. *Fatou and Julia sets.* Every holomorphic $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map, i.e. it can be written in the form

$$f(z) = \frac{p(z)}{q(z)}$$

for complex polynomials p, q with no common factors. The *degree* of f is the maximum of the degrees of p and q, and is equal to the number of preimages of any $w \in \hat{\mathbb{C}}$ that is not a critical value (i.e. not of the form $f(z)$ for $f'(z) = 0$). We shall restrict attention in the sequel to the case that $d \geq 2$.

Let f be a rational map. We denote by f^n the map obtained by composing f with itself n times; thus $f^1 = f$ and if f has degree d, then f^n has degree d^n.

Definition 1.7 (Fatou and Julia sets)**.** Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map, and let $\mathcal{F}$ be the family consisting of f and all its iterates; i.e. $\mathcal{F} := \{f^n : n \in \mathbb{N}\}$. The *Fatou set* $F(f)$ is the maximal open subset of $\hat{\mathbb{C}}$ on which $\mathcal{F}$ is normal. The *Julia set* $J(f)$ is the complement of the Fatou set; i.e. $J(f) := \hat{\mathbb{C}} - F(f)$.

Thus, a point z is in $F(f)$ if and only if $\mathcal{F}$ is normal in some neighborhood of z. Equivalently, $z \in J(f)$ if and only if no infinite subsequence of iterates of f converges uniformly on $\bar{U}$ for any neighborhood U of z.

A set $X \subset \hat{\mathbb{C}}$ is *completely invariant* if $f(X) = X$ and $f^{-1}(X) = X$. Such sets can be characterized in the following way. Given $x \in \hat{\mathbb{C}}$ the *grand orbit* of x, denoted $[x]$, is the set of $y \in \hat{\mathbb{C}}$ for which there are integers n and m with $f^n(x) = f^m(y)$. Equivalently, if we let $\sim$ be the smallest equivalence relation for which $z \sim f(z)$ for every $z \in \hat{\mathbb{C}}$, then the grand orbit of x is the equivalence class of x under $\sim$. One sees immediately that $X \subset \hat{\mathbb{C}}$ is completely invariant if and only if it is a union of grand orbits. Hence (for instance) X is completely invariant if and only if $\hat{\mathbb{C}} - X$ is.

Furthermore, if X is completely invariant, then so is its closure $\bar{X}$ and its derived set X' (i.e. its set of accumulation points).

Example 1.8 (Exceptional points). A point x is *exceptional* if $[x]$ is finite (and therefore every element of $[x]$ is exceptional too). It turns out that there are at most 2 exceptional points. For, otherwise, let E be a finite set of exceptional orbits of cardinality > 2. Then $X := \hat{\mathbb{C}} - E$ is hyperbolic and therefore $f : X \to X$ is 1-Lipschitz and therefore area non-increasing for the hyperbolic metric (which is finite). But this contradicts the fact that the degree $d \geq 2$.

It is rare for f to have any exceptional points. We may always conjugate f by an automorphism of $\hat{\mathbb{C}}$ so that one of the exceptional points is ∞ and the other (if it exists) is at 0. Thus: either f is conjugate to a polynomial, or to a map of the form $z \to z^d$ for some $d \in \mathbb{Z}$ with $|d| \geq 2$. Notice in every case that the exceptional points (if any) are in $F(f)$.

Lemma 1.9. *The sets $J(f)$ and $F(f)$ are completely invariant.*

Proof. It suffices to prove this for $F(f)$, which amounts to showing that $\mathcal{F}$ is normal at z if and only if it is normal at $f(z)$. But this is obvious: f takes some small neighborhood U of z to some small neighborhood $f(U)$ of z by a branched cover of some degree, and evidently $\mathcal{F}$ is normal on U if and only if it is normal on $f(U)$. □

Now, because the degree of f^n is d^n, the family $\mathcal{F}$ can't be equicontinuous on all of $\mathbb{CP}^1$; thus $J(f)$ is nonempty and, since it is completely invariant, it must be infinite.

Lemma 1.10. *Let E be a closed completely invariant set. Then either $|E| \leq 2$ or $J(f) \subset E$. In other words, $J(f)$ is the minimal closed completely invariant subset with at least 3 points.*

Proof. If $|E| > 2$ the complement $\Omega := \hat{\mathbb{C}} - E$, which is open and completely invariant, is also hyperbolic, so $\mathcal{F}|\Omega$ is normal, and $\Omega \subset F(f)$. □

Lemma 1.11. *$J(f)$ is perfect.*

Proof. If X is closed and completely invariant, then so is its derived set X' (i.e. the set of limit points of X). Since $J(f)$ is infinite, its derived set is nonempty. If it were finite, it would be exceptional and therefore in $F(f)$. Thus $J(f)$ is equal to its derived set; i.e. it is perfect. □

Lemma 1.12. *Let f and g be rational maps that commute. Then $J(f) = J(g)$. In particular $J(f^n) = J(f)$ for all n.*

Proof. Let $z \in F(f)$. Then $\mathcal{F}$ is equicontinuous on some neighborhood U of z so $g\mathcal{F}$ is equicontinuous on U so $\mathcal{F}$ is equicontinuous on $g(U)$. Thus $F(f)$ is g-invariant, and since it is open and its complement contains at least three points, $F(f) \subset F(g)$ and by symmetry we are done. □

1.2.2. *Periodic Orbits.* A *periodic orbit of order* n is a finite set of distinct points z_i for $i = 0, \cdots, n-1$ so that $f(z_i) = z_{i+1}$ for all i, indices taken mod n. For each i the chain rule says

$$(f^n)'(z_i) = \prod_{j=0}^{n-1} f'(z_j)$$

We call this common value the *multiplier*, and denote it μ.

We say a periodic orbit is

(1) *superattracting* if $\mu = 0$ (equivalently if some z_i in the orbit is critical for f);
(2) *attracting* if $0 < |\mu| < 1$;
(3) *indifferent* if $|\mu| = 1$; these are further distinguished into those that are
  (a) *rationally indifferent* if μ is a root of unity; and
  (b) *irrationally indifferent* if μ is not a root of unity;

 and
(4) *repelling* if $|\mu| > 1$.

Lemma 1.13. *Every attracting and superattracting periodic orbit is in $F(f)$.*

Proof. If U is a sufficiently small open neighborhood of z_0 (say), then $f^n(U) \subset U$ and therefore $V := \cup_n f^n(U)$ omits at least three points. But then $\mathcal{F}|U$ is normal. □

Conversely,

Lemma 1.14. *Every repelling periodic orbit is in $J(f)$.*

Proof. The derivatives $(f^{mn})'(z_0) = \mu^m$ are unbounded in norm, so $\mathcal{F}$ is not equicontinuous on any neighborhood of z_0. □

Indifferent periodic orbits might be in either the Julia or the Fatou set. However, we have the following:

Lemma 1.15. *Every rationally indifferent periodic orbit is in $J(f)$.*

Proof. After a change of coordinates we may let 0 be a rationally indifferent periodic point, and then some power of f can be written near 0 in the form

$$f^n(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

Since the degree of f is bigger than 1, there is some first non-zero coefficient a_m; i.e. $f^n(z) = z + a_m z^m + o(z^m)$, and then $f^{kn}(z) = z + k a_m z^m + o(z^m)$ so that the mth derivatives of f^{kn} are not uniformly bounded near 0, so that 0 is in $J(f)$. □

Lemma 1.16. *$J(f)$ is contained in the closure of the set of periodic orbits.*

Proof. After replacing f by an iterate of f if necessary, we may assume the degree $d \geq 3$. Let $z \in J(f)$ and let U be a small neighborhood of z. Since $J(f)$ is perfect, we may assume z is not a critical value for f. Thus there are at least three distinct preimages of z under f, say z_1, z_2, z_3 all distinct from z, contained in neighborhoods U_1, U_2, U_3 with disjoint closures all mapped homeomorphically to U by f^2.

For each $w \in U$ let $w_i \in U_i$ be the corresponding preimage of w, and let $\phi_w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the unique Möbius transformation taking w_1, w_2, w_3 to $\infty, 0, 1$. Evidently ϕ_w are an equicontinuous family, depending analytically on w. For each n define $g_n : U \to \hat{\mathbb{C}}$ by $g_n(w) := \phi_w f^n(w)$, and let $\mathcal{G} := \{g_n\}$. Then $\mathcal{G}$ is normal on U if and only if $\mathcal{F}$ is. But by definition of $J(f)$, the family $\mathcal{F}$ is not normal on any neighborhood of any point in $J(f)$, and therefore $\mathcal{G}$ is not normal on U. It follows that there is some g_n and some $w \in U$ for which $g_n(w) \in \{\infty, 0, 1\}$; equivalently, $f^n(w) = w_i$ for some i. But then $f^{n+1}(w) = w$ so w is in a periodic orbit, and $J(f)$ is contained in the closure of the set of periodic orbits as claimed. □

1.2.3. *Attracting basins.* Let $O := \{z_0, \cdots, z_{n-1}\}$ be an attracting or superattracting periodic orbit. The *attracting basin* $B(O)$ of O is the set of points z so that $f^n(z)$ is eventually contained in any open neighborhood U of O. Evidently $B(O)$ is open; the connected components containing O form the *immediate basin.* Note that if O, O' are distinct periodic orbits then $B(O)$ and $B(O')$ are disjoint.

Lemma 1.17. *Let $B(O)$ be the attracting basin of a (super)-attracting periodic orbit O. Then*

(1) $B(O)$ is totally invariant;
(2) $B(O) \subset F(f)$;
(3) Components of $B(O)$ are components of $F(f)$.

Proof. That $B(O)$ is totally invariant is immediate from the definition.

Let U be a hyperbolic open neighborhood of O with $f(U) \subset U$. Every $z \in B(O)$ has a neighborhood V which is mapped into U by some f^n; thus every sufficiently large iterate of f maps V into U, so $V \subset F(f)$ by Montel.

Let A be a component of $B(O)$ contained in a component B of $F(f)$. Sufficiently big iterates of f^n converge uniformly on compact subsets of A to the constant map to some $z_j \in O$. Since B is connected and contained in $F(f)$, these iterates must have a subsequence that converges uniformly on compact subsets of B to the constant map to z_j. Thus $B \subset B(O)$. □

Lemma 1.18. *Let O be an attracting or superattracting periodic orbit. Then the immediate basin of O contains some critical point. Hence there are at most $2d-2$ attracting or superattracting periodic points.*

Proof. If O is superattracting there is nothing to prove, so we suppose O is merely attracting. Further, by replacing f by an iterate if necessary, we may assume O is an attracting fixed point z of f. Let U be the immediate basin of z. We shall prove that U contains a critical point.

Since U is equal to the component of $F(f)$ containing z, the map $f : U \to U$ is proper. If U contained no critical points, then $f : U \to U$ would be a covering map and therefore a (local) isometry in the hyperbolic metric. Thus $|f'|$ would equal 1 at any fixed point, contrary to the fact that $|f'(z)| < 1$ at the attracting fixed point z. □

1.3. **Indifferent periodic orbits.**

1.3.1. *Fatou's Lemma.* Let N_a denote the number of attracting or superattracting periodic orbits, and let N_i denote the number of indifferent periodic orbits. We have shown $N_a \le 2d-2$. In fact, Shishikura [14] showed by the method of quasiconformal surgery that $N_a + N_i \le 2d-2$, but a more elementary argument (due to Fatou, that we shall give shortly) gives the weaker estimate $N_a + N_i/2 \le 2d-2$. Fatou shows that one may perturb f to a nearby rational function f_w in such a way that *at least half* of the indifferent periodic points of f are perturbed to attracting periodic points off_w. If the deformation is small enough, the (super)attracting periodic points of f are approximated by (super)attracting periodic points of f_w, and therefore the estimate follows.

The perturbations Fatou considers are of the form

$$f_w(z) := (1-w)f(z) + w$$

where $w = \rho e^{i\theta}$ for some suitably chosen $\theta \in [0, 2\pi]$, and any ρ is real, positive, and sufficiently small.

Lemma 1.19 (Fatou)**.** *For some w as above, at least half the indifferent periodic points of f are perturbed to attracting periodic points of f_w. Consequently $N_a + N_i/2 \le 2d - 2$.*

Proof. This proof is from Blanchard [8], Thm. 5.12.

Write $F(z, w) := f_w(z) = (1-w)f(z) + w$ and for each indifferent periodic point z_i with period n_i and multiplier s_i, write $G_i(z, w) := F^{n_i}(z, w) = z$. Let $V_i \subset \mathbb{C}^2$ be the variety defined by $G_i(z, w) = 0$. This is a variety because G_i is a rational function, so by clearing denominators, the zero locus of G_i is equal to the zero locus of some polynomial.

V_i is 1-dimensional and not contained in the line $w = 0$ the projection of any sheet (near $(z_i, 0)$) to the w-line is locally an m_i-fold branched cover. Thus (again, locally) on (some sheet of) V_i we can write z_i as an analytic function of an m_ith root $w_i := w^{1/m_i}$. In other words, for all sufficiently small w_i, the point $z_i(w_i)$ is a periodic point of $F(\cdot, w_i^{m_i})$ with period n_i. Define

$$s_i(w_i) := \frac{\partial F^{n_i}}{\partial z}(z_i(w_i), w_i^{m_i})$$

i.e. $s_i(w_i)$ is the multiplier of the periodic point $z_i(w_i)$ of $F(\cdot, w_i^{m_i})$.

Choose a finite collection of indifferent periodic orbits (a posteriori we can choose all of them), and let m be the least common multiple of the m_i, and let $v^m = w$ so that $v^{m/m_i} = w_i$. Then for each i we can expand $s_i(v)$ as

$$s_i(v) = s_i + a_i v^{k_i} + \cdots$$

Note that some (least) coefficient a_i is nonzero, because as we analytically continue to $w = 1$ the map f_w converges uniformly (away from the poles of f) to the constant function $f_1(z) = 1$. We claim we can find an arbitrarily small v for which $|s_i + a_i v^{k_i}| < 1$ for at least half the indices i; this will complete the proof.

If we fix θ and let $v = \rho e^{i\theta}$ for small positive ρ then for each z_i we have $\arg(a_i v^{k_i}) = \arg(a_i) + k_i\theta$ so the set of angles θ for which this argument points strictly inside the unit circle at s_i has measure $1/2$ (where the circle is normalized to have measure 1). Thus there is a θ for which at least half these arguments simultaneously point strictly inside the unit circle at their respective s_i, and we are done. □

Theorem 1.20 (Periodic Julia)**.** *The Julia set $J(f)$ is the closure of the set of repelling periodic orbits.*

Proof. We have shown that $J(f)$ is contained in the closure of the set of all periodic orbits. Since $J(f)$ is perfect, and there are only finitely many (super)-attracting or indifferent periodic orbits, it follows that $J(f)$ is contained in the closure of the set of repelling periodic orbits. But every repelling periodic orbit is in $J(f)$. □

Lemma 1.21. *Let U be any open set intersecting $J(f)$. Then $\hat{\mathbb{C}} - \cup f^n(U)$ is contained in the set of exceptional points. Furthermore, for all sufficiently large n the image $f^n(U)$ contains $J(f)$.*

Proof. If $V := \cup f^n(U)$ omits three points then $\mathcal{F}$ is normal on U, contrary to definition. Thus $\hat{\mathbb{C}} - V$ contains at most two points. If either of these points is not exceptional, it has an infinite backward orbit which meets V, thus this point is in V after all. This proves the first claim.

Now let $p \in U$ be a periodic repelling point. By replacing U by an even smaller open set if necessary we may assume $U \subset f^n(U)$. By the previous argument there is m (a multiple of n) with $J(f) \subset f^m(U)$. But then $J(f) = f^k J(f) \subset f^{m+k}(U)$ for all k. $\square$

1.3.2. *Petals.* Let O be a rationally indifferent periodic orbit, and for simplicity, after conjugating by translation, and replacing f by a power if necessary, we can assume $f(0) = 0$ and $f'(0) = 1$. Thus we can write $f(z) = z + az^{m+1} + \cdots$ for some $m \geq 1$. Conjugating by $z \to \alpha z$ for $\alpha^m = a$ puts f in the form $f(z) = z + z^{m+1} + \cdots$. For f in this form we have the following lemma:

Lemma 1.22. *Let* 0 *be a rationally indifferent fixed point of* f *with multiplier* 1*, and suppose near* 0 f *has the form* $z \to z(1 + z^m + o(z^m))$*. Then there are* m petals Π_k *homeomorphic to closed disks, each containing* 0*, and satisfying*

1. *the* Π_k *are pairwise disjoint except at* 0*;*
2. $\partial \Pi_k$ *is real analytic except at* 0 *where it has a corner, and is tangent to the rays* $\arg(z) = 2\pi k/m$ *and* $\arg(z) = 2\pi(k+1)/m$*;*
3. f *maps each* Π_k *inside itself, and* f^n *converges uniformly to the constant function to* 0 *on* Π_k*;*
4. $\arg(f^n(z))$ *converges to* $\pi(2k+1)/m$ *uniformly on compact subsets of* $\Pi_k - 0$*; and*
5. $|f(z)| < |z|$ *on a neighborhood of the* axis $\arg(z) = \pi(2k+1)/m$ *in* Π_k*.*

Proof. We claim that f is analytically conjugate on some neighborhood of 0 to something of the form
$$f(z) = z(1 + z^m + bz^{2m} + cz^{2m+1} + \cdots)$$
To see this, observe that the map $z \to z + bz^{r+1}/(m-r)$ conjugates $z + z^{m+1} + bz^{m+r+1} + \cdots$ to $z + z^{m+1} + b'z^{m+r+2} + \cdots$ for some b', and apply induction.

For each residue k mod m let S_k denote the open sector of the unit disk where $\arg(z) \in (2\pi k/m, 2\pi(k+1)/m)$ and let $\sigma : S_k \to \mathbb{C}$ be the map $\sigma(z) = 1/z^m$. The image is the open slit domain $W \subset \mathbb{C}$ consisting of z with $|z| > 1$ and z not positive real. Now define $g(z) := \sigma f \sigma^{-1}$ which is well-defined and analytic on W. In terms of a local coordinate w on W we can write
$$g(w) = w - m + a/w + o(w^{-1})$$
for a suitable constant a, and where $o(w^{-1})$ means a term arbitrarily small compared to $|w^{-1}|$ when $|w|$ is big.

Let $P_t \subset W$ be the region $y^2 > 4t(x + t)$ bounded by a parabola. For sufficiently large t, the map g takes P_t properly inside itself. Furthermore, g is topologically conjugate to a translation on P_t, and in fact converges to the translation $w \to w - m$ where $|w| \gg 1$.

The preimage of P_t in S_k is the petal Π_k. From the properties of g the lemma follows. $\square$

1.3.3. *Parabolic components.*

Definition 1.23. An f-invariant component U of $F(f)$ is *parabolic* (or a *Leau domain*) if there is a rationally indifferent fixed point z in ∂U and if f^n converges on U to the constant map to z.

Lemma 1.24. *Every petal is contained in a parabolic component. Conversely, if U is a parabolic component with rationally indifferent fixed point $z \in \partial U$, and Π_k are petals for z, there is a unique Π_k in U, and for every $z \in U$ we have $f^n(z) \in \Pi_k$ for sufficiently large n.*

Proof. After conjugation and passing to a power of f if necessary we may assume our rationally indifferent fixed point is 0 and $f(z) = z + az^{m+1} + \cdots$. Note that $f^{-1}(z) = z - az^{m+1} + \cdots$ is also rationally indifferent on a neighborhood of 0 (where it is defined). Let Π_k be petals for f, and for sufficiently small t, let S'_k be sectors of the disk $D_t(0)$ (contained in petals for f^{-1} and contained in a neighborhood of the axes of these petals) so that $|f^{-1}(z)| < |z|$ throughout each S'_k and so that $D_t(0) \subset \cup \Pi_k \cup S'_k$.

If U is the component of $F(f)$ containing Π_k then since f^n converges to 0 on Π_k the same is true on U (by normality); thus U is parabolic.

Conversely, for any parabolic component containing 0, and for $z \in U$ there is some n so that $0 < |f^{n+1}(z)| < |f^n(z)| < t$ so that $f^n(z)$ is not in any S'_j, and is therefore in Π_j for some j. In other words, for every $z \in U$ we have $f^n(z) \in \Pi_j$ for some j and for all sufficiently large n. But the Π_k are disjoint and forward-invariant, so a parabolic domain can contain at most one of them. □

Parallel to Lemma 1.18 we have

Lemma 1.25. *Every parabolic cycle contains a critical point.*

Proof. By replacing f by an iterate if necessary we may assume U is an f-invariant parabolic domain. Since $f : U \to U$ is proper, if U contained no critical points, it would be a covering map hence a local isometry in the hyperbolic metric.

But in fact we claim that for any compact $K \subset U$ the hyperbolic diameter of $f^n(K)$ converges to infinity. To see this we fix coordinates and notation as in Lemma 1.22 and let $V \subset U$ by taken by σ to a half-space $H := \{z : \text{real}(z) < C << 0\}$ and such that σ conjugates f on V to $g : w \to w - m + a/w + o(w^{-1})$ on H. Every compact $K \subset U$ has an iterate that lands in V, and by the Schwarz Lemma the hyperbolic metric on V dominates the restriction of the hyperbolic metric on U, so we just need to show that for any compact $K' \subset H$ the diameter of $g^n(K)$ in the hyperbolic metric on H goes to zero. But g asymptotically preserves the Euclidean metric deep in H, and the ratio of the Euclidean to the hyperbolic metric in H is equal to the (Euclidean) distance to ∂H. The claim follows. □

1.4. **Siegel disks and Herman rings.**

1.4.1. *Siegel disks.* Let O be an irrationally indifferent periodic orbit, and again after conjugation and replacing f by an iterate if necessary, let's suppose $f(0) = 0$ and $f'(0) = \mu$ where $\mu = e^{2\pi i\alpha}$ for some irrational α.

Lemma 1.26. *The following are equivalent for 0 an irrationally indifferent fixed point of f with multiplier μ.*

(1) 0 is in the Fatou set $F(f)$;
(2) f is linearizable at 0 — i.e. it is analytically conjugate to $z \to \mu z$ on a neighborhood of 0;

(3) $0 \in F(f)$*, and the component* U *of* $F(f)$ *containing* 0 *is an open disk on which* f *is analytically conjugate to* $z \to \mu z$ *on* $\mathbb{D}$*.*

Proof. Evidently (2) implies (1), and (3) implies both (1) and (2), so we just show that (1) implies (3).

The map f takes U to itself and lifts to the universal cover $\tilde{f} : \tilde{U} \to \tilde{U}$ fixing some lift $\tilde{0}$ of 0. Since U is hyperbolic, $\tilde{U}$ is isomorphic to $\mathbb{D}$; and since the derivative of $\tilde{f}$ at $\tilde{0}$ is μ, by the Schwarz Lemma, $\tilde{f}$ on $\tilde{U}$ is analytically conjugate to an irrational rotation $z \to \mu z$ on $\mathbb{D}$. In particular, $f : U \to U$ is a local isometry in the hyperbolic metric.

We claim $U = \tilde{U}$; i.e. U is simply-connected. For if not, there is some finite set of shortest nontrivial geodesics from 0 to itself, and this set would be taken to itself by f, contrary to the fact that α is irrational. □

Definition 1.27. An irrationally indifferent fixed point z is called a *Siegel point* if f is linearizable at z, and a *Cremer point* otherwise. A component of $F(f)$ containing a Siegel point is called a *Siegel disk*.

There are some irrational α for which there exist both Siegel and Cremer points (of different f) with multiplier $e^{2\pi i\alpha}$. However, Cremer [10] and Siegel [15] gave sufficient conditions in terms of α alone to guarantee that the point is a Cremer point resp. a Siegel point.

Theorem 1.28 (Cremer [10])**.** *Let* z *be an irrationally indifferent fixed point of* f *of degree* d *with multiplier* $e^{2\pi i\alpha}$*. If there is a sequence of* n *for which the* d^n*th roots of* $|1-\mu^n|$ *converge to* 0 *then* f *is not linearizable at* z*.*

Proof. After conjugation by a Möbius transformation we can put the indifferent fixed point at 0, and (because this is a simple fixed point) we can arrange for $f(\infty) = 0$. It follows that we can write f as a rational function $p(z)/q(z)$ where the degree of q is d and the degree of p is strictly less than d. Conjugation by a dilation $z \to \lambda z$ multiplies the z^j coefficient of a polynomial by λ^{j-1} so we can arrange for p and q to be of the form

$$p(z) = \mu z + a_2 z^2 + \cdots + a_{d-1} z^{d-1}, \quad q(z) = 1 + b_1 z + \cdots + z^d$$

For any n we can write f^n in the form $f^n(z) = p_n(z)/q_n(z)$ where

$$p_n(z) = \mu^n z + \cdots + c_n z^{d^n - 1}, \quad q_n(z) = 1 + \cdots + z^{d^n}$$

A periodic point of f of order n is a solution of $f^n(z) = z$ which is a root of a polynomial of the form

$$zq_n(z) - p_n(z) = z(z^{d^n} + \cdots + (1-\mu^n))$$

The product of the nonzero roots of this polynomial is $\pm(1-\mu^n)$; since there are d^n of them, there is at least one root of absolute value less than or equal to the d^nth root of $|1-\mu^n|$. Thus under the stated conditions on μ there are periodic points of f arbitrarily close to z, so that f is not linearizable at z. □

Note that the set of irrational $\alpha \in [0,1]$ satisfying the hypothesis of Cremer's theorem is the intersection of a countable collection of open dense sets. Thus it is residual in the sense of the Baire category theorem.

A much deeper theorem is due to Siegel:

Theorem 1.29 (Siegel [15]). *Let z be an irrationally indifferent fixed point of f with multiplier $e^{2\pi i\alpha}$. Suppose there are positive constants $a, b > 0$ so that $|\alpha - p/q| > a/q^b$ for all $p, q \in \mathbb{Z}$ with $q \geq 1$. Then f is linearizable at z.*

Proof. This proof is due to Carleson and Gamelin [9], Thm. II.6.4. Without loss of generality we let the fixed point be 0. We want to find an injective holomorphic map ϕ, defined in a neighborhood U of 0, with $\phi(0) = 0$ and $\phi'(0) = 1$ and solving

$$\phi^{-1}f\phi(z) - \mu z = 0 \tag{1.1}$$

For any function of the form $\xi(z) := \lambda z + O(z^2)$ we will write $\bar{\xi}(z) := \xi(z) - \lambda z$; i.e. $\bar{\xi}$ denotes the part of ξ of order ≥ 2. Thus $\bar{f}(z) = f(z) - \mu z$, likewise $\bar{\phi}(z) = \phi(z) - z$ and so on. With this notation Equation 1.1 becomes

$$\bar{f}(\phi(z)) = \phi(\mu z) - \mu\phi(z) = \bar{\phi}(\mu z) - \mu\bar{\phi}(z) \tag{1.2}$$

and we are trying to find ϕ for which the conjugate $g := \phi^{-1}f\phi$ satisfies $\bar{g} = 0$.

We find such a ϕ iteratively and we will consider that we have made progress if $\bar{g}$ is 'smaller than' $\bar{f}$ in a suitable sense. The nonlinear term in the argument of $\bar{f}$ makes it hard to solve Equation 1.2 directly, so the iteration procedure is to have ϕ solve the simpler equation

$$\bar{\phi}(\mu z) - \mu\bar{\phi}(z) = \bar{f}(z) \tag{1.3}$$

For such a ϕ we have

$$\bar{g}(z) + \bar{\phi}(\mu z + \bar{g}(z)) = \mu\bar{\phi}(z) + \bar{f}(z + \bar{\phi}(z))$$

and therefore by Equation 1.3

$$\bar{g}(z) = \bar{\phi}(\mu z) - \bar{\phi}(\mu z + \bar{g}(z)) + \bar{f}(z + \bar{\phi}(z)) - \bar{f}(z) \tag{1.4}$$

If we write $f(z) = \mu z + \sum_{n>1} b_n z^n$ then the solution to Equation 1.3 is

$$\bar{\phi}(z) = \sum_{n>1} \frac{b_n}{\mu^n - \mu} z^n \tag{1.5}$$

The diophantine condition $|\alpha - p/q| > a/q^b$ is equivalent to $|\mu^n - 1| > cn^{-\beta}$ for some c and for $\beta = b - 1$. At the cost of changing the constant c, let's rewrite this as

$$\frac{1}{|\mu^n - 1|} < \frac{cn^\beta}{\beta!}$$

Furthermore, let's suppose there are constants $\delta > 0$ and $r > 0$ for which we have estimates of the form

$$|\bar{f}'(z)| \leq \delta \text{ for } |z| < r \tag{1.6}$$

from which it follows by Cauchy's estimate that $|b_n| \leq \delta/nr^{n-1}$. Note that by choosing r small enough we can assume δ is as small as we like because $\bar{f}(z) = O(z^2)$, so we can certainly find small ϵ so that $c\delta < \epsilon^{\beta+2}$ and $\delta < \epsilon$.

Let's estimate $|\bar{g}'(z)|$. From Equation 1.5 we have

$$|\bar{\phi}'(z)| \leq \sum_{n>0} \frac{n|b_n|}{|\mu^n - \mu|}|z|^{n-1} \leq \frac{c\delta}{\beta!} \sum n^\beta (|z|/r)^n$$

so we have

$$|\bar{\phi}'(z)| \le \frac{c\delta}{\beta!}\sum n^\beta(1-\epsilon)^n \le \frac{c\delta}{\epsilon^{\beta+1}} \text{ for } |z| < (1-\epsilon)r \tag{1.7}$$

so if $c\delta < \epsilon^{\beta+2}$ then $|\bar{\phi}'| \le \epsilon$ in the disk of radius $(1-\epsilon)r$. From this and $|\bar{f}'| \le \delta < \epsilon$ one easily sees that ψ maps the disk of radius $(1-4\epsilon)r$ into the disk of radius $(1-3\epsilon)r$, and g maps the disk of radius $(1-4\epsilon)r$ into the disk of radius $(1-\epsilon)r$.

Let D denote the disk of radius $(1-4\epsilon)r$ and E the disk of radius $(1-\epsilon)r$, and for a function on either disk denote its maximum by $|\cdot|_D$ or $|\cdot|_E$ respectively. From Equation 1.4 it follows that

$$|\bar{g}|_D \le |\bar{\phi}'|_E|\bar{g}|_D + |\bar{f}'|_E|\bar{\phi}|_D \le \epsilon|\bar{g}|_D + \delta|\bar{\phi}|_D \tag{1.8}$$

By Equation 1.7 the map $\bar{\phi}'$ takes the disk of radius $(1-\epsilon)r$ to the disk of radius $c\delta/\epsilon^{\beta+1}$. Furthermore it vanishes at the origin. Thus by the Schwarz Lemma

$$|\bar{\phi}'(z)| \le \frac{|z|c\delta}{(1-\epsilon)r\epsilon^{\beta+1}}$$

throughout E so integrating gives $|\bar{\phi}|_D \le (1/2)c\delta(1-3\epsilon)r/\epsilon^{\beta+1}$. Substituting in Equation 1.8 gives $|\bar{g}|_D \le (1/2)c\delta^2 r/\epsilon^{\beta+1}$ and therefore by Cauchy's estimate (applied to balls of radius $r\epsilon$ completely contained in D) we obtain the inequality

$$|\bar{g}'(z)| \le \frac{1}{2}\frac{c\delta^2}{\epsilon^{\beta+2}} \le \frac{1}{2}\delta \text{ for } |z| < (1-5\epsilon)r \tag{1.9}$$

Comparing this to Equation 1.6 we see that the bound on the derivative has gone down by a factor of 2 after one iteration, at the cost of reducing the radius by a factor of $(1-5\epsilon)$ subject only to the inequalities $c\delta < \epsilon^{\beta+2}$ and $\delta < \epsilon$. So we can perform the substitution $\delta \to \delta/2$ and $\epsilon \to \epsilon/2^{1/(\beta+2)}$ and iterate. Thus we obtain a sequence of (holomorphic) conjugacies from f to functions g_n that converge uniformly to $z \to \mu z$ on some disk of radius $r\prod(1-5\epsilon/2^{n/(\beta+2)}) > 0$. □

The $\alpha \in [0,1]$ satisfying the hypothesis of Theorem 1.29 have full measure. We shall state (but not prove) a sharpening of these results due to Brjuno and Yoccoz in § 3.2.

1.4.2. *Herman rings.*

Definition 1.30. A *Herman ring* is an annular component U of $F(f)$ on which f is holomorphically conjugate to an irrational rotation.

Herman rings can occur; we shall see some examples in the sequel. But not for polynomials:

Lemma 1.31. *Let f be a polynomial. Then $F(f)$ contains no Herman rings.*

Proof. Let γ be an f-invariant circle which is the core of a Herman ring, on which f acts as an irrational rotation. Let D be the disk bounded by γ not containing ∞. Since f is a polynomial, $f(D)$ is bounded, and by the maximum modulus principle $f(D) = D$. Since f has degree 1 on ∂D it has degree 1 on D; i.e. $f : D \to D$ is an isomorphism. But then D is contained in a Siegel disk. □

Siegel disks and Herman rings do not contain critical points, so they admit no direct analog of Lemma 1.18 or Lemma 1.25. However one does have the following:

Lemma 1.32. *If U is a Siegel disk or Herman ring then ∂U is contained in the closure of the forward images of the critical points.*

Proof. The map f is invertible on U. Denote its inverse on U by $g := (f|U)^{-1}$. Suppose there is $z \in \partial U$ contained in a small disk D that does not contain a forward image of any critical point. Thus g and (by induction) all its iterates extend to D, and $f^n g^n$ is equal to the identity on D for all n.

We claim that $\mathcal{G} := \{g^n\}$ is normal on D. To see this, let A and B be two disjoint cycles for f, each with at least 3 points. If $z \in D - A$ then no $g^n(z) \in A$ or else $f^n g^n(z) \in A \cap (D - A) = \emptyset$, so $\mathcal{G}$ is normal in $D - A$, and for the same reason in $D - B$, thus in D.

Now, $g|U$ is an irrational rotation, so there is some sequence of iterates g^{n_i} that converges uniformly to the identity on U. It follows that a further subsequence converges uniformly to the identity on D. Thus there is some smaller disk $z \in E \subset D$ and a subsequence of iterates $f^{n_i}(E) \subset D$. But this contradicts Lemma 1.21. □

1.5. Classification of invariant components.

Theorem 1.33 (Classification of Invariant Components)**.** *Let U be an invariant component of $F(f)$. Then U is one of the following:*

(1) a super-attracting component;
(2) an attracting component;
(3) a parabolic component;
(4) a Siegel disk; or
(5) a Herman ring.

All five possibilities can occure for rational maps f. We shall prove this theorem shortly, but first we prove some lemmas.

Let U be forward invariant, and suppose $\mathcal{G}$ is the set of analytic functions on U that are (locally uniform) limits of subsequences of $f^n|U$.

Lemma 1.34. *Suppose $g \in \mathcal{G}$ is the constant function to some ζ. Then $\zeta \in \overline{U}$ and ζ is fixed by f.*

Proof. Since U is f-invariant, any limit must take U to $\overline{U}$ proving the first claim. By definition of $\mathcal{G}$, there is convergence $f^{n_i}(z) \to \zeta$ for some sequence n_i, locally uniform in U. But then

$$f(\zeta) = f(\lim f^{n_i}(z)) = \lim f^{n_i}(f(z)) = \zeta$$

□

Lemma 1.35. *Suppose $g \in \mathcal{G}$ is the constant function to $\zeta \in U$. Then ζ is attracting or super-attracting and U is an attracting or super-attracting component.*

Proof. If $\zeta \in U$ then there is $\zeta \in V \subset U$ with $f^{n_i}(\overline{V}) \subset V$ for some n_i. But then by the Schwarz Lemma $|(f^{n_i})'(\zeta)| < 1$. □

Lemma 1.36. *Suppose every $g \in \mathcal{G}$ is constant. Then $\mathcal{G}$ consists of exactly one function, and therefore f^n converges locally uniformly in U to some $\zeta \in \overline{U}$.*

Proof. Suppose $\mathcal{G}$ consists of the constant functions to some ζ_j, necessarily all in ∂U (or we could apply the previous lemma). Note that every ζ_j is necessarily indifferent, so there are only finitely many ζ_j, and we choose finitely many disjoint open neighborhoods V_j of the ζ_j.

Let $K \subset U$ be compact, and by enlarging it if necessary suppose $K \cap f(K)$ is nonempty, so that $L := \cup f^n(K)$ is connected. Then $f^n(L)$ is eventually contained in $\cup V_j$ or else we could find an element of $\mathcal{G}$ converging on K to some new ζ'. But L is connected, so $f^n(L) \subset V_j$ for some specific j, and therefore $f^m(K) \subset V_j$ for all $m \geq n$. This shows $\mathcal{G}$ consists only of the constant function to a single $\zeta = \zeta_j$, and therefore f^n converges locally uniformly on U to ζ. □

Lemma 1.37. *Suppose f^n converges locally uniformly on U to $\zeta \in \partial U$. Then ζ is rationally indifferent, and U is a parabolic component.*

Proof. By conjugation we can assume $\zeta = 0$, and let μ be the multiplier. Note $|\mu| = 1$.

Let W be the interior of the set $f^n(L)$ defined in the previous lemma, so that $W \subset U$ is connected and forward invariant, and contained in a neighborhood V of 0 where f is injective. Fix $w \in W$ and define $\phi_n(z) := f^n(z)/f^n(w)$ for $z \in W$. We claim $\{\phi_n\}$ is a normal family in W. Since f is injective in W, no function ϕ_n takes the values 0, 1 or ∞ on $W - z$, so $\{\phi_n\}$ is certainly normal on $W - w$. To show it is normal near w, let D be a small round disk around w. Then $\{\phi_n\}$ is normal in a neighborhood of ∂D so there is a subsequence ϕ_{n_j} that converges on ∂D. Thus $|\phi_{n_j}|$ is uniformly bounded on ∂D and by the maximum principle the same is true on D; this proves the claim.

Thus ϕ_n contains some subsequence that converges locally uniformly on W to a limit ϕ which evidently satisfies $\phi(f(z)) = \mu\phi(z)$. Because the ϕ_n are injective, the limit ϕ is either injective or constant. Since $\phi(w) = 1$, if ϕ is constant it is equal to 1 everywhere, which shows $\mu = 1$.

Otherwise ϕ is injective, so there is a small round disk D around w for which $|\phi(z) - \phi(w)| > \epsilon$ when $z \in W - D$. But $\phi(f^n(w)) = \mu^n\phi(w) = \mu^n$ so because $|\mu| = 1$ there are n_i with $\phi(f^{n_i}(z)) \to 1$. Furthermore, $f^{n_i}(z) \to 0$ so we obtain a contradiction, and see that ϕ is constant after all. □

We now give the proof of Theorem 1.33.

Proof. The theorem is proved (and we are in one of cases (1)–(3)) unless $\mathcal{G}$ contains a non-constant function g. We first show $g(U) \subset U$. To see this, let $w \in U$ be arbitrary and let $D \subset U$ be a small closed disk around w for which $g(z) - g(w)$ has no zeroes on ∂D. By the definition of g there is n for which $f^n(z) - g(w)$ is arbitrarily close to $g(z) - g(w)$ on ∂D and therefore by Rouché's theorem, $f^n(z) - g(w)$ and $g(z) - g(w)$ have the same (finite, positive) number of zeros on D. Thus there is $z \in D$ with $f^n(z) = g(w)$ so that $g(U) \subset U$.

Now suppose n_i are such that $f^{n_i} \to g$, and by passing to a further subsequence if necessary, we can assume that $m_i := n_{i+1} - n_i$ increase without bound. There is a further subsequence so that $f^{m_i} \to h \in \mathcal{G}$ and then

$$hg(z) = \lim f^{m_i}(f^{n_i}(z)) = \lim f^{n_{i+1}}(z) = g(z)$$

Since g is non-constant, h must be the identity on $g(U)$ and therefore on all of U. This implies that $f : U \to U$ is both injective and surjective, or else h couldn't be.

It follows that f is an isometry of U in its hyperbolic metric, and furthermore that f^n has a subsequence converging to the identity. This implies that U is conformally either an annulus or a disk, and f is (in either case) conjugate to an irrational rotation. □

2. No Wandering Domains

Let f be a rational map of the Riemann sphere of degree $d \geq 2$. A component U of the Fatou set is said to be *eventually periodic* if there is a nonnegative integer m so that $f^m(U)$ is periodic (i.e. there is a positive integer n so that $f^{n+m}(U) = f^m(U)$). Eventually periodic components are essentially classified by Theorem 1.33. A component U which is not eventually periodic (i.e. such that the components $f^n(U)$ are all distinct) is said to be *wandering*.

Thus the final piece in the classification theorem for components of $F(f)$ is Sullivan's celebrated

Theorem 2.1 (Sullivan; No Wandering Domains)**.** *Let f be a rational map of $\hat{\mathbb{C}}$. Then every component of $F(f)$ is eventually periodic.*

This is [17], Thm. 1. We give a streamlined proof written down by Zakeri using substantial simplifications due to Baker and McMullen.

2.1. Reduction to simply-connected domains.

Lemma 2.2 (Baker)**.** *Suppose $U \subset F(f)$ is a wandering domain. Then $f^n(U)$ is simply connected for $n \gg 1$.*

Proof. Let $U_n := f^n(U)$ and for the sake of argument let $\infty \in U$. Since f has only finitely many critical points, by replacing U by some U_n if necessary, we may assume that each $f^n : U \to U_n$ is a covering map. Since the U_n are disjoint, the spherical areas $\text{area}(U_n) \to 0$. Since $\mathcal{F}|U$ is normal, any limit is constant, so the spherical diameters $\text{diam}(f^n(K)) \to 0$ for all compact $K \subset U$.

Let $\gamma \subset U$ be any loop, and let $\gamma_n = f^n(\gamma)$. This might be immersed; let B_n be the subset of $\mathbb{C}$ bounded by γ_n. Then $\text{diam}(\gamma_n) \to 0$ so also $\text{diam}(B_n) \to 0$, and because $\hat{\mathbb{C}}$ is compact, $\text{diam}(f(B_n)) \to 0$ in the spherical metric. But since $\partial f(B_n) \subset \gamma_{n+1} \subset U_{n+1} \subset \mathbb{C} - U$ it follows that $f(B_n)$ does not contain ∞ for big n. Thus eventually $f(B_n) \subset B_{n+1}$ so $\mathcal{F}|B_n$ have range disjoint from some neighborhood of infinity. But then $\mathcal{F}|B_n$ is normal, so $B_n \subset F(f)$ and therefore γ_n is null-homotopic in U_n. Since $f : U \to U_n$ is a covering map it follows that γ is null-homotopic in U. Since γ was arbitrary, U is simply-connected and the lemma is proved. □

2.2. Quasiconformal deformations.

Sullivan's theorem depends on the theory of quasiconformal deformations, as developed chiefly by Ahlfors and Bers in the 1960s. We give a very brief introduction to this subject, as we will use it extensively in the sequel. For details see e.g. Ahlfors [1].

If $z := x + iy$ is a local holomorphic coordinate on a Riemann surface S then we have $dz := dx + idy$ and $d\bar{z} := dx - idy$. These complex-valued smooth 1-forms are dual to (complex valued) vector fields $\partial_z := (1/2)(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} := (1/2)(\partial_x + i\partial_y)$. Here 'vector fields' can be interpreted as (complex valued) derivations on smooth functions on S. We also write ∂ for ∂_z and $\bar{\partial}$ for $\partial_{\bar{z}}$ and for a smooth complex-valued function φ we write $\varphi_z = \partial_z\varphi = \partial\varphi$ and $\varphi_{\bar{z}} = \partial_{\bar{z}}\varphi = \bar{\partial}\varphi$. The Cauchy-Riemann equations say that φ is holomorphic iff $\varphi_{\bar{z}} = 0$.

Define the *Beltrami differential* μ_φ (or just μ if φ is understood) to be the differential form

$$\mu := \mu(z)\frac{d\bar{z}}{dz} = \frac{\varphi_{\bar{z}}d\bar{z}}{\varphi_z dz}$$

In terms of a different local holomorphic coordinate w we have

$$\mu(z)\frac{d\bar{z}}{dz} = \mu(w)\frac{d\bar{w}}{dw} \text{ so } \mu(z) = \mu(w)e^{2i\theta}$$

where θ is the argument of dz/dw. Thus $|\mu(z)|$ is independent of a choice of local holomorphic coordinate.

The Jacobian of φ satisfies $J(f) = |\varphi_z|^2 - |\varphi_{\bar{z}}|^2$. If φ is an orientation-preserving diffeomorphism on a domain U then $|\mu| < 1$ there.

Example 2.3. Any real linear map from $\mathbb{C}$ to $\mathbb{C}$ has the form $T : z \to \alpha z + \beta\bar{z}$ for unique complex numbers α, β. For such a map $\mu = \beta/\alpha$ so we could alternately write $T : z \to \alpha(z + \mu\bar{z})$; i.e. T is the composition of the 'stretch' map $z \to z + \mu\bar{z}$ with a dilation.

For any smooth orientation-preserving diffeomorphism φ with Beltrami differential μ, the image of an infinitesimal round circle at p is an infinitesimal ellipse at $\varphi(p)$ whose major and minor axes have lengths in the ratio $K(p) := (1 + |\mu(p)|)/(1 - |\mu(p)|)$. The function K is also called the *dilatation* of φ.

Definition 2.4 (Quasiconformal Diffeomorphism)**.** If φ is an orientation-preserving diffeomorphism on some domain with Beltrami differential μ, then if the supremum K of $(1 + |\mu|)/(1 - |\mu|)$ on the domain is finite, we say φ is K-*quasiconformal*.

It is important to extend this definition to orientation-preserving homeomorphisms which are not necessarily smooth.

Definition 2.5 (Quasiconformal Homeomorphism)**.** A *quadrilateral* Q in an open domain U is a subset homeomorphic to a closed disk, together with a choice of four points in ∂Q called the *vertices*. The *modulus* of Q, denoted $K(Q) \ge 1$, is the ratio of the edge lengths of a Euclidean rectangle R for which there is a homeomorphism $Q \to R$, conformal in the interior, and taking vertices to vertices.

An orientation-preserving homeomorphism $\varphi : U \to \mathbb{C}$ is K-*quasiconformal* if, for every quadrilateral $Q \subset U$ with modulus 1, the image $\varphi(Q)$ has modulus $\le K$. It is *quasiconformal* on U if it is K-quasiconformal for some K.

Quasiconformal maps enjoy the following properties:

1. **(Locality):** A map φ is K-quasiconformal on U iff for every $p \in U$ it is K-quasiconformal on some open neighborhood of p in U.
2. **(ACL):** A K-quasiconformal map φ is *absolutely continuous on lines*. This means for every rectangle $R \subset U$, for almost every horizontal (or vertical) line $I \subset R$, the real and imaginary parts of $\varphi = f + ig$ are absolutely continuous on I; i.e. for all

$\epsilon > 0$ there is a $\delta > 0$ so that if $[a_j, b_j] \subset I$ are a finite set of disjoint intervals with total length $< \delta$, then $\sum |f(b_i) - f(a_i)| < \epsilon$ and similarly for g.

(3) **(Beltrami differential):** If φ is K-quasiconformal on U, the derivatives φ_z and $\varphi_{\bar{z}}$ are defined a.e. in the sense of distribution, and $\mu_\varphi := \varphi_{\bar{z}}/\varphi_z$ is a measurable, essentially bounded complex-valued function on U. Furthermore,
$$\frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} \leq K$$
(4) **(Regularity):** A smooth map is K-quasiconformal in the sense of Definition 2.4 if and only if it is K-quasiconformal in the sense of Definition 2.5. An orientation-preserving homeomorphism is conformal (and therefore smooth) if and only if it is 1-quasiconformal.

See e.g. Ahlfors [1] or Lehto [12] for proofs.

Definition 2.6. A *Beltrami differential* is a differential $\mu := \mu(z)d\bar{z}/dz$ on a domain $U \subset \mathbb{C}$ where $\mu(z)$ is measurable, and the ess. sup. $\|\mu\|_\infty$ is finite.

Since $|\mu(z)|$ is independent of the choice of local holomorphic coordinate, we may define Beltrami differentials on any Riemann surface S. The space of Beltrami differentials on S is a complex Banach space with respect to the ess. sup. norm, and is denoted $B(S)$.

The *measurable Riemann mapping theorem* (see [2] for a proof) says the following:

Theorem 2.7 (Measurable Riemann mapping theorem)**.** *Let $\mu \in B(\hat{\mathbb{C}})$ be a Beltrami differential on $\hat{\mathbb{C}}$ with $\|\mu\|_\infty < 1$. Then there is a* unique *orientation-preserving quasiconformal homeomorphism $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\mu_\varphi = \mu$ and fixing $0, 1, \infty$ (one says such a φ is* normalized*). Furthermore, for each z the value of $\varphi(z)$ depends holomorphically on μ.*

The existence and uniquenss of φ is due to Morrey; the holomorphic dependence on parameters is due to Ahlfors-Bers.

Beltrami differentials pull back under holomorphic maps and push forward under holomorphic isomorphisms. Thus, if U is a wandering domain for f, and $f^n : U \to U_n$ is an isomorphism for all positive n, we may construct an f-invariant Beltrami differential μ (i.e. $f^*\mu = \mu$) on $\hat{\mathbb{C}}$ by defining it however we like on U, pulling it back to preimages $f^{-n}U$ and pushing it forward to U_n, and then extending it by 0 on the rest of $\hat{\mathbb{C}}$. Let $B(\hat{\mathbb{C}})^f$ denote the f-invariant Beltrami differentials on $\hat{\mathbb{C}}$.

If μ is f-invariant then so is $t\mu$ for all $t \in [0, \infty)$. For each t for which $\|t\mu\|_\infty < 1$ let φ_t be the normalized quasiconformal homeomorphism of $\hat{\mathbb{C}}$ associated to $t\mu$ by Theorem 2.7, and let $f_t := \varphi_t f \varphi_t^{-1}$. Then by the chain rule, f_t is 1-quasiconformal and is therefore a rational map of the same degree as f. Furthermore, the map $t \to f_t$ is holomorphic in t (by Ahlfors-Bers), and its derivative at zero $\dot{f}$ therefore lies in $T_f\mathrm{Rat}_d$, the (holomorphic) tangent space at f to the space Rat_d of rational maps of degree d. Summarizing, we get a sequence of linear maps
$$B(U) \to B(\hat{\mathbb{C}})^f \to T_f\mathrm{Rat}_d$$
2.3. **Completion of the proof.** The idea of the proof is now rather easy to explain. The space $B(U)$ is evidently infinite dimensional, whereas $T_f\mathrm{Rat}_d$ is finite dimensional (it has complex dimension $2d - 2$). The proof of Theorem 2.1 will therefore be completed if we

can show that the existence of a wandering domain U implies the existence of a subspace of $B(U)$ whose image in $T_f\mathrm{Rat}_d$ has arbitrarily large dimension.

The first step is to analyze Beltrami differentials for which $\dot{f} = 0$. Let v be the vector field on $\hat{\mathbb{C}}$ obtained by differentiating φ_t; i.e. $v(z) := d/dt|_{t=0}\varphi_t(z)$ for all z. Because the $\varphi_t(z)$ are normalized, $\varphi_t(z) = z + tv(z) + o(t)$ and therefore $v_{\bar{z}} = \mu$.

Lemma 2.8. *Suppose $\dot{f} = 0$. Then v vanishes on $J(f)$.*

Proof. We compute

$$\dot{f} = \frac{d}{dt}|_{t=0}f_t(z) = \frac{d}{dt}|_{t=0}\varphi_t f \varphi_t^{-1} = v(f(z)) - f'(z)v(z)$$

Therefore $\dot{f} = 0$ if and only if $v(f(z)) = f'(z)v(z)$ for all z. But this implies that v vanishes on any periodic orbit with multiplier not equal to 1, in particular on any repelling periodic cycle. Since periodic repelling cycles are dense in $J(f)$, the lemma follows. □

This Lemma is the infinitesimal analog of Sullivan [17], Prop. 5 which says that the group of homeomorphisms of $J(f)$ commuting with f is totally disconnected (this is obvious because such a homeomorphism must permute the finite set of points that are periodic with period diving any fixed n; thus the homeomorphism group injects into an infinite product of finite permutation groups). A quasiconformal map φ conjugating f to itself must induce a homeomorphism of $J(f)$ to itself, commuting with f; thus a 1-parameter family φ_t for which $f_t = f$ must fix $J(f)$ pointwise.

Any *compactly supported* quasiconformal homeomorphism of U extends to an f-equivariant homeomorphism of $\hat{\mathbb{C}}$ that commutes with f; thus we need to find (families of) homeomorphism(s) that act nontrivially on ∂U. Now, U is simply-connected, and (because $J(f)$ has more than two points) is holomorphically isomorphic to the open unit disk $\mathbb{D}$. If $\mu \in B(\mathbb{D})$ has $\|\mu\|_\infty < 1$ we may extend μ outside $\mathbb{D}$ however we like (without increasing the norm) and solve the Beltrami equation, obtaining φ with $\varphi_{\bar{z}}/\varphi_z = \mu$ on $\mathbb{D}$. Postcomposing with a holomorphic isomorphism $\varphi(\mathbb{D}) \to \mathbb{D}$ gives rise to a quasiconformal homeomorphism $\phi : \mathbb{D} \to \mathbb{D}$ with $\phi_{\bar{z}}/\phi_z = \mu$. Any family G of real analytic diffeomorphisms of $\partial\mathbb{D}$ are the boundary values of some family G' of (real analytic) quasiconformal diffeomorphisms of $\mathbb{D}$ associated in this way to Beltrami differentials in $B(\mathbb{D})$; identifying $B(\mathbb{D})$ with $B(U)$ therefore gives rise to a family of quasiconformal homeomorphisms of $\hat{\mathbb{C}}$ that act nontrivially on ∂U (which may be compared with $\partial\mathbb{D}$ via Caratheodory's theory of prime ends). So choosing G of dimension $> 2d - 2$ completes Sullivan's argument; see [17], § 9.

McMullen gave an elegant infinitesimal version of this argument that avoids the use of prime ends. Here it is.

Let $N' \subset B(\mathbb{D})$ be the space of Beltrami differentials on $\mathbb{D}$ of the form $p(\bar{z})d\bar{z}/dz$ for some polynomial p. The vector field

$$v_k(z) := \bar{z}^{k+1}\partial_z \text{ on } \mathbb{D}, \quad v_k(z) = z^{-(k+1)}\partial_z \text{ on } \hat{\mathbb{C}} - \mathbb{D}$$

is quasiconformal on $\hat{\mathbb{C}}$ with $(v_k)_{\bar{z}} = (k+1)\bar{z}^k d\bar{z}/dz$ on $\mathbb{D}$ and zero on $\hat{\mathbb{C}} - \mathbb{D}$. Therefore, if V denotes the linear span of the v_k, for any $\mu \in N'$ there is a unique $v \in V$ with $v_{\bar{z}} = \mu$ on $\mathbb{D}$. On the other hand, if w is any other vector field on $\mathbb{D}$ with $w_{\bar{z}} = \mu$ on $\mathbb{D}$ and $w|\partial\mathbb{D} = 0$ then $v - w$ is holomorphic on $\mathbb{D}$ and agrees with v on $\partial\mathbb{D}$. But from the form of the v_k, the unique meromorphic extension to $\mathbb{D}$ of $v|\partial\mathbb{D}$ has a pole at zero unless $v = 0$ in which case also $\mu = 0$.

Now let $N \subset B(\mathbb{D})$ be the space of Beltrami differentials that agree with some element of N' on the disk of radius $1/2$ and are zero elsewhere. If $v_{\bar{z}} = \mu$ on $\mathbb{D}$ and $v|\partial\mathbb{D} = 0$ for $\mu \in N$ then v is holomorphic on the annulus $1/2 < |z| \leq 1$ and since $v|\partial\mathbb{D} = 0$ we must have v identically zero on this annulus, and hence $v = 0$ on $|z| = 1/2$. Reasoning as above, $\mu = 0$. In short:

Lemma 2.9. *There is an infinite dimensional space $N \subset B(\mathbb{D})$ of Beltrami differentials μ compactly supported in the interior of $\mathbb{D}$ so that if $v_{\bar{z}} = \mu$ on $\mathbb{D}$ and $v|\partial\mathbb{D} = 0$ then $\mu = 0$.*

Now let $\psi : U \to \mathbb{D}$ be a conformal isomorphism, and define $N(U) := \psi^* N$ for N as above. Let $\mu \in N(U)$ and let v be a quasiconformal vector field on $\hat{\mathbb{C}}$ with $v_{\bar{z}} = \mu$ on U and $v|\partial U = 0$. Then $\psi(v)$ is a vector field on $\mathbb{D}$ which is holomorphic near $\partial\mathbb{D}$ and converges to 0 as $|z| \to 1$; thus by the reflection principle, $\psi(v)$ is identically zero near $\partial\mathbb{D}$. But $(\psi(v))_{\bar{z}} = (\psi^{-1})^*\mu \in N$ so $(\psi^{-1})^*\mu = 0$ so $\mu = 0$.

It follows that $N(U)$ maps injectively to $T_f\mathrm{Rat}_d$, which is absurd. This completes the proof.

3. Examples

3.1. Attracting and Super-attracting fixed points.

Example 3.1. The simplest rational map (of degree $d > 1$) is $z \to z^d$. The points 0 and ∞ are superattracting, with (totally invariant) attracting basins equal to the open unit disk, and the exterior of the closed unit disk respectively. The Julia set is the unit circle, and $z \to z^d$ takes J to itself by an expanding d-fold covering map.

Example 3.2. Let f be a perturbation of the previous example: $f : z \to z^d + p(z)$ where $p(z)$ is a polynomial of degree $< d$ with coefficients of size $o(1)$. Then f and $z \to z^d$ are very close away from a small neighborhood of 0. The point ∞ is still superattracting, but (at least for generic $p(z)$) the superattracting point 0 is perturbed into an ordinary attracting point z_0, the unique root of $z^d + p(z) - z$ near zero.

Because $z \to z^d$ is uniformly expanding on S^1, the dynamics there is structurally stable, so the Julia set $J(f)$ is a topological circle on which f is conjugate to $z \to z^d$. Another way to see this is to use quasiconformal surgery.

Let α be the circle $|z| = 2^{-1}$. Let β be its preimage under $z \to z^d$ and let β_f be its preimage under f. Thus β is the circle $|z| = 2^{-1/d}$ and β_f is a real analytic simple closed curve very near to β. Let D be the disk bounded by β and let D_f be the disk bounded by β_f. Let $\phi : D_f \to D$ be a diffeomorphism which is the identity on the (common) subdisk bounded by α, and such that $(\phi(z))^d = \phi(f(z)) = f(z)$ for $z \in \beta_f$. Now define a new map $G : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as follows:

$$G(z) = \begin{cases} f(z) \text{ for } z \in \hat{\mathbb{C}} - D_f \\ (\phi(z))^d \text{ for } z \in D_f \end{cases}$$

The map G agrees with f outside D_f, agrees with $z \to z^d$ inside α, and is smooth (and therefore K-quasiconformal for some K) on the annulus A_f between β_f and α. Let μ_0 be the Beltrami differential $f_{\bar{z}}d\bar{z}/f_z dz$ on A_f, and let μ be the Beltrami differential which is equal to μ_0 on A_f, and to $(f^n)^*\mu_0$ on the annulus $f^{-n}(A_f)$. These annuli are disjoint, and each is taken to the next by a d-fold cover under f. If φ solves the Beltrami equation for μ, then φ conjugates G to a degree d rational map with two superattracting fixed points of order d; in other words, φ conjugates G (up to a Möbius transformation) to $z \to z^d$. Since G and f are holomorphically conjugate on a neighborhood of $J(f)$, it follows that $J(f)$ is a quasicircle — the image of a round circle under a quasiconformal map; and that f is conjugate to $z \to z^d$ on S^1 there. This construction is essentially due to Douady–Hubbard.

3.2. **Indifferent fixed points.**

Example 3.3. The map $f : z \to z^2 + e^{2\pi i\theta}z$ for θ real has an indifferent fixed point at 0. When $\theta = p/q$ is rational, this is a rationally indifferent fixed point, so it is in $J(f)$ on the boundary of a cycle of q petals, each contained in parabolic components.

For example, the map $z \to z^2 + z$ is conjugate to $z \to z^2 + 1/4$ which has an indifferent fixed point at $1/2$. Real numbers (slightly) less than $1/2$ are in the parabolic component, and converge to $1/2$; real numbers greater than $1/2$ are in the basin of infinity and diverge away from it. The petal has a 'cusp' at $1/2$, so that for any w which is not real and positive, $1/2 + \epsilon w$ is in the petal and converges to $1/2$ for $\epsilon > 0$ real and sufficiently small.

Recall that an irrationally indifferent fixed point z for f is called a *Siegel point* if f is (holomorphically) linearizable near z, and a *Cremer point* if not.

Example 3.4. When θ is an irrational number satisfying Siegel's criterion, $f : z \to z^2+e^{2\pi i\theta}z$ is linearizable at 0, so there is a Siegel disk around 0 on which f is (holomorphically) conjugate to an irrational rotation through angle θ. Any irrational which is not too well approximated by rational numbers satisfies Siegel's criterion; for example θ equal to the golden ratio $(1 + \sqrt{5})/2$.

Example 3.5. If p_n/q_n are the successive continued fraction approximations to θ, then Cremer's theorem says that $f : z \to e^{2\pi i\theta} + O(z^2)$ is *not* linearizable if $\sup \log q_{n+1}/q_n = \infty$. Brjuno improved Siegel's theorem to show that if $\sum \log q_{n+1}/q_n < \infty$ then f is linearizable, and Yoccoz [18] showed that Brjuno's condition is sharp: the quadratic map $f : z \to z^2 + e^{2\pi i\theta}z$ is linearizable at zero if and only if Brjuno's condition holds.

3.3. **Herman rings.**

Polynomial maps cannot have Herman rings. Examples of rational maps with Herman rings may be constructed by quasiconformal surgery.

Example 3.6. Let f be a map with an invariant Siegel disk D and rotation number θ. Let B be an f-invariant closed subdisk of D, and let $A \subset D$ be an f-invariant closed annular neighborhood of ∂B that is split by ∂B into A^+ outside B and A^- inside B.

Let $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasiconformal homeomorphism, conformal outside A^-, taking $\hat{\mathbb{C}} - B$ conformally to the upper half plane H^+ and B quasiconformally to the lower half plane H^-, and so that $\phi(A^-)$ is contained in the image under complex conjugation of $\phi(A^+)$. Let E be the annulus $\phi(A^+)$ together with its complex conjugate.

Now define a new map $G : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $G(z) = \phi f \phi^{-1}(z)$ for $z \in H^+$, and defining G in H^- to be the reflection of F in H^+; i.e. $G(\bar{z})$ is defined to be the complex conjugate of $G(z)$ for $z \in H^+$.

Notice that the annulus E is G-invariant, and G acts *holomorphically* on E, conjugate to rotation by θ. Define $E_1 := F^{-1}(E) - E$. Then G is quasiconformal on E_1, and conformal on $\hat{\mathbb{C}} - E_1$. Inductively define $E_n := G^{-1}(E_{n-1})$. Then the E_n are disjoint, and each (for $n > 1$) is the full G-preimage of E_{n-1}. Let μ_0 be the Beltrami differential of G on E_1 and let μ be the Beltrami differential on $\hat{\mathbb{C}}$ that is $(G^{-(n-1)})^*\mu_0$ on each E_n and 0 outside $\cup_n E_n$. The solution φ to the Beltrami equation with differential μ is holomorphic on E and conjugates G to a rational map g of degree $2\deg(f) - 1$. The annulus $\varphi(E)$ is contained in a Herman ring for g.

This construction is essentially due to Shishikura [14], § 9. He showed moreover that a rational map with a Herman ring must have degree at least 3.

Example 3.7. The rational map $f : z \to (e^{2\pi i\theta} z^2(z-4))/(1-4z)$ where $\theta = 0.6151732\ldots$ has a Herman ring on which f acts as rotation by the golden ratio. This example was found by Shishikura, by computer experiment.

3.4. **Smooth curves in** $J(f)$**.**

Example 3.8. The map $z \to z^2$ has $J = S^1$. The map $z \to 2z^2 - 1$ has $J = [-1, 1]$. These maps are actually semiconjugate: the map $\varphi : z \to (z + 1/z)/2$ takes $\hat{\mathbb{C}} - [-1, 1]$ to $\hat{\mathbb{C}} - \overline{\mathbb{D}}$ and (semi)conjugates $z \to 2z^2 - 1$ to $z \to z^2$.

Example 3.9 (Blaschke Products). By the Schwarz Lemma, every analytic automorphism of the unit disk is of the form

$$z \to e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

where $a \in \mathbb{D}$ is the (unique) preimage of 0. Any such automorphism takes S^1 to itself with degree 1. The product (*not* the composition!) of finitely many automorphisms therefore takes $\mathbb{D}$ properly to itself by a degree n map. Such a product is called a *Blaschke Product*; for $a := a_1, \cdots, a_n$ an unordered collection of n (not necessarily distinct) points in $\mathbb{D}$, and for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the associated Blaschke product is the function

$$B : z \to e^{i\theta} \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}$$

Conversely, if $f : \mathbb{D} \to \mathbb{D}$ is proper of degree n with zeros at a the ratio B/f is holomorphic and nowhere zero on $\bar{\mathbb{D}}$, and has absolute value 1 on S^1 and is therefore equal to the constant function to $e^{i\theta}$ for some θ.

Not every Blaschke product fixes a point in the interior of $\mathbb{D}$. A degree n rational map has $n+1$ fixed points (counted with multiplicity). The map $B : S^1 \to S^1$ has degree n so if $|B'(z)| > 1$ on S^1 then B has exactly $n-1$ fixed points on S^1 and one each in $\mathbb{D}$ and $\hat{\mathbb{C}} - \overline{\mathbb{D}}$ (if B is uniformly expanding on S^1 then it takes a sufficiently large compact round subdisk of $\mathbb{D}$ properly inside itself and thus has a fixed point there). The unique fixed point $z \in \mathbb{D}$ is necessarily (super)-attracting; thus $\mathbb{D}$ is precisely equal to the basin of attraction of z.

If $|B'(z)| < 1$ somewhere on S^1 then $e^{i\theta}B$ has $n+1$ fixed points on S^1 for suitable θ and therefore none in $\mathbb{D}$. If $|B'(z)| \geq 1$ on S^1 and $|B'(z)| = 1$ at some $z \in S^1$ then $e^{t\theta}B$ has a (rationally indifferent) parabolic fixed point on S^1 for suitable θ, and $\mathbb{D}$ is a parabolic component.

Example 3.10. If f has an invariant component $U \subset F(f)$ homeomorphic to $\mathbb{D}$, and $\phi : U \to \mathbb{D}$ is a uniformizing map, then $B := \phi f \phi^{-1}$ is a Blaschke product on $\mathbb{D}$. Suppose U

is a (super)-attracting component, so that after conjugacy we may assume 0 is a (super)-attracting fixed point for B.

Let C be any other Blaschke product of the same degree as B, also with a fixed point in $\mathbb{D}$. Then B and C are quasiconformally conjugate on suitable neighborhoods of $\partial\mathbb{D}$ so if $\psi : \mathbb{D} \to \mathbb{D}$ is such a quasiconformal conjugacy, we may define f_C to be equal to $\phi\psi^{-1}C\psi\phi$ on U and to f elsewhere. There is a Beltrami differential μ_U on U so that if φ_U solves the Beltrami equation for μ_U on U, then φ_U conjugates f_C to a holomorphic map on U (which is holomorphically conjugate to C on $\mathbb{D}$). Let μ be obtained by iteratively pulling back μ_U to $f^{-n}(U) - U$ and extending by 0 elsewhere, and let φ solve the Beltrami equation for μ on $\hat{\mathbb{C}}$. Then φ conjugates f to a new rational map with an invariant component $\varphi(U)$ on which the new map is holomorphically conjugate to C.

3.5. **Examples with $J(f) = \hat{\mathbb{C}}$.** The following classical example is due to Lattès (1918).

Example 3.11 (Lattès example). Let E be an elliptic curve, which may be uniformized as $\mathbb{C}/\Lambda$ for some lattice Λ. Let $\eta \in \mathbb{C}^*$ be such that $\eta\Lambda \subset \Lambda$; for example, we could take η to be any nonzero integer. The map $\eta : z \to \eta z$ on $\mathbb{C}$ descends to a degree $d := |\eta|^2$ self-covering map $g : E \to E$ so that if $d > 1$, repelling periodic orbits for g are dense in E. On the other hand, multiplication by η on E commutes with multiplication by -1, which is an involution ι of E with fixed points at the four points $(1/2)\Lambda/\Lambda$. The quotient E/ι is a genus 0 Riemann surface, so that multiplication by η on E descends to a degree d endomorphism of E/ι, still with the property that repelling periodic orbits are dense. The Weierstrass $\wp$ function is even, and uniformizes E/ι as the Riemann sphere. Thus $\wp$ semi-conjugates g to a degree d rational map f with $J(f)$ equal to the whole Riemann sphere. Such an f is known as a Lattès map.

Notice that for any lattice Λ and any $n \in \mathbb{Z}$ we have $n\Lambda \subset \Lambda$. Thus Lattès maps of this kind (which have degree n^2) come in (one complex dimensional) families. Real affine linear automorphisms of $\mathbb{C}$ take any Λ to any other, and commute with multiplication by n and -1; thus these flexible Lattès families are all quasiconformally conjugate, and the conjugating maps have f-invariant Beltrami differentials, necessarily supported on $J(f)$. This is in contrast to the case of Kleinian groups, for which Sullivan [16] famously proved that quasiconformal deformations of Kleinian groups are quasiconformally rigid on their limit sets.

Let's write down an explicit example of f. Let E be the 'square' elliptic curve $\mathbb{C}/(Z+i\mathbb{Z})$ and let $\eta = 1+i$. A fundamental domain for ι is the rectangle R with real part in $[0, 1/2]$ and imaginary part in $[-1/2, 1/2]$. The four fixed points of ι are 0, $1/2$, $\pm i/2$ and $1/2 \pm i/2$. There are two (simple) critical points of g, at $(1 \pm i)/4$. Thus E/ι may be made by gluing two squares (of side length 1/2) along their boundaries in such a way that the fixed point of ι are the vertices, and the critical points of g are the centers of the two squares. In other words, conformally the fixed points of ι and the critical points of g are the vertices of a regular octahedron. By composing $\wp$ with a Möbius map we may obtain ϕ uniformizing this Riemann surface as the Riemann sphere in such a way that

$$\phi : 0, (1+i)/2, i/2, 1/2 \to \infty, 0, i, -i \text{ and } \phi : (1+i)/4, (1-i)/4 \to -1, 1$$

This conjugates g to a degree 2 rational map f with critical points at $1, -1$ that map by

$$1, -1 \to -i, i \to 0 \to \infty \to \infty$$

Thus $f(z) = (z^2+1)/(2iz)$.

Notice that for this example, the critical points are preperiodic but not periodic. It turns out that this condition alone implies $J(f) = \hat{\mathbb{C}}$:

Lemma 3.12. *Suppose every critical point of f is preperiodic but not periodic. Then $J(f) = \hat{\mathbb{C}}$.*

Proof. A superattracting cycle has a periodic critical point. Every attracting or parabolic cycle contains the forward image of a critical point which is not preperiodic. The boundary of every Siegel disk or Herman ring is contained in the closure of the set of forward iterates of (necessarily non-preperiodic) critical points. There are no wandering domains. Thus $F(f)$ is empty. □

3.6. **Dendrites.** A *dendrite* is a subset of $\hat{\mathbb{C}}$ that is closed and connected with connected complement and empty interior. Parallel to Lemma 3.12 we have

Lemma 3.13. *Let f be a polynomial so that every critical point of f except ∞ is strictly preperiodic. Then $J(f)$ is a dendrite.*

Proof. Arguing as in Lemma 3.12 we have that $F(f)$ is equal to the immediate basin U of ∞. We claim U is a disk (this will prove the lemma since then $J(f) = \partial F(f) = \partial U$ will be connected). The map f is a d-fold cover of $U - \infty$ to itself, and is therefore injective on π_1. Furthermore, for any compact $K \subset U - \infty$ the iterates $f^n(K)$ are eventually contained in any neighborhood of ∞. It follows that π_1 is abelian, so that $U - \infty$ is an annulus so that U is a disk. □

Example 3.14. For the map $z \to z^2 + i$ the unique finite critical point 0 has orbit

$$0 \to i \to i - 1 \leftrightarrow -i$$

3.7. **Wandering domains.**

Example 3.15. A rational map does not have a wandering domain, by Sullivan's theorem. But an entire holomorphic map $f : \mathbb{C} \to \mathbb{C}$ *can* have one. The entire function $g(z) := z - \lambda \sin(2\pi z)$ for small real positive λ has attracting fixed points at every integer n. Let U_n be the Fatou component containing n. A Fatou component (even for a transcendental function) certainly can't contain more than one attracting fixed point; thus the U_n are disjoint.

Now consider $f(z) := z - \lambda \sin(2\pi z) + 1$. This is the composition of g with the translation $\tau(z) := z + 1$. Since g commutes with τ, both g and f commute with τ and with each other. The following Lemma is due to Baker [4] Lemma 4.5:

Lemma 3.16. *If f and g are entire with $g = f + c$ for some constant c, and if f and g commute, then $J(f) = J(g)$.*

Proof. Baker shows [3] (by a careful analysis of the argument of Lemma 1.16) that for f entire, $J(f)$ is equal to the closure of the set of repelling periodic orbits. Since f and g commute, g takes an f-periodic orbit to an f-periodic orbit of the same or smaller period. Since $f' = g'$ such an orbit has the same f-multiplier as its g-image. Thus $g(J(f)) \subset J(f)$.

Let $z \in F(f)$ and suppose the iterates of f have a subsequence that converges uniformly on a neighborhood U of z to some function h. Since $g(f^n(U)) = f^n(g(U))$ it follows that if $h(U)$ is bounded, some sequence of iterates of f are bounded on $g(U)$, so $g(U) \subset F(f)$.

If there is no neighborhood U of z on which f^n has a subsequence which is bounded, then actually there is convergence $f^n(z) \to \infty$ and because $z \in F(f)$, we can find a neighborhood U of z and an n_0 so that $|f^n(w)| \gg 1$ for all $w \in U$ and all $n \geq n_0$. But then $f^n(g(U)) = g(f^n(U)) = f^{n+1}(U) + c$ avoids a neighborhood of 0 for all sufficiently large n so by Montel's theorem, $g(U) \subset F(f)$. Thus we have shown $g(F(f)) \subset F(f)$ so in fact $g(J(f)) = J(f)$.

But then $J(f)$ is totally invariant for g, so $J(g) \subset J(f)$ and by symmetry $J(g) = J(f)$. □

Hence for our example the U_n are distinct components of the Fatou set of f. Since $f(U_n) = U_{n+1}$, it follows that each U_n is wandering for f. Baker attributes this kind of example to Herman.

4. Post-critically finite maps

Let $f : S^2 \to S^2$ be an orientation-preserving branched covering, that is, f is locally a diffeomorphism away from finitely many points (the *critical points* $C(f)$) where f is smoothly conjugate to $z \to z^q$ for some $q > 1$ (the *local degree* of f at the given critical point).

Definition 4.1. An orientation-preserving branched covering f is *post-critically finite* if every critical point has a finite orbit. In other words, if the set $P(f) := \cup_{c \in C} \cup_{n>0} f^n(c)$ is finite.

The degree d of f is defined in the usual way.

Definition 4.2. Two post-critically finite maps f, g are *equivalent* if there are homeomorphisms $\theta, \theta' : (S^2, P(f)) \to (S^2, P(g))$ isotopic rel. $P(f)$, so that $\theta f = g\theta'$ as maps from $(S^2, P(f))$ to $(S^2, P(g))$.

The goal of this section is to give a necessary and sufficient criterion, due to Thurston, for a post-critically finite f to be equivalent to a rational map (Theorem 4.6). Such a rational map will necessarily be unique up to (holomorphic) conjugacy, with one exceptional family to be explained in the sequel. In fact, Thurston's argument does more than prove the existence of an equivalent rational map — it gives a (convergent) algorithm to find it.

The basic idea is rather simple. An isotopy class of conformal structure on S^2 rel. $P(f)$ pulls back under f to an isotopy class of conformal structure on S^2 rel. $f^{-1}P(f)$, and thereby (since $P(f) \subset f^{-1}P(f)$) an isotopy class of conformal structure on S^2 rel. $P(f)$. Thus, pullback under f defines a holomorphic map σ_f from a certain Teichmüller space $\mathcal{T}_f$ to itself, and a rational map equivalent to f is the same thing as a fixed point for this map.

Any holomorphic endomorphism of a Teichmüller space is distance non-increasing in the Teichmüller metric, and is strictly distance decreasing except under rather special circumstances, and therefore uniformly distance contracting on compact subsets (morally, this is a manifestation of the Schwarz Lemma). The special circumstances are ruled out whenever P is sufficiently complicated (f has a 'hyperbolic orbifold') and then the orbit of

σ_f stays in a compact subset of Teichmüller space and consequently has a (unique) fixed point unless a topological condition holds, known as a 'Thurston obstruction'.

Thurston's account of his theorem is mostly unwritten; a detailed exposition was given by Douady–Hubbard [11] and we follow their paper.

Example 4.3. Suppose f of degree 2 has one totally invariant point (which we can take to be ∞) and one finite critical point c which is periodic of order 3. Without loss of generality let's take $c = 0$ and then $f(0) = v$, $f'(0) = 0$, $f(v) = 1$, $f(1) = 0$. To specify f topologically we must make some choices, so let's suppose v is real and negative, and f takes the interval $[v, 1]$ to itself by the piecewise linear map that takes $[0, 1]$ to $[v, 0]$ preserving orientation and $[v, 0]$ to $[v, w]$ reversing orientation.

The point v is the unique finite critical value, so f^{-1} is defined on a 2-fold cover of $\hat{\mathbb{C}}$ branched at ∞ and v, i.e. on the Riemann surface of $z \to \lambda\sqrt{z - v}$ for any $\lambda \in \mathbb{C}^*$. If we normalize by the choice $\lambda := 1/\sqrt{-v}$ then f^{-1} pulls back

$$v, 0, 1 \to 0, 1, \frac{-\sqrt{1-v}}{\sqrt{-v}}$$

We can interpret this pullback as a map on Teichmüller space $\mathcal{T}_f$. With respect to the normalization that f leaves ∞ totally invariant, that 0 is the unique finite critical point with image v, and that $f(v) = 1$ and $f(1) = 0$, the space $\mathcal{T}_f$ is parameterized by a lift of v to the universal cover of $\mathbb{C} - \{0, 1\}$. Thus the map σ_f acts on the negative real axis by

$$v \to \frac{-\sqrt{1-v}}{\sqrt{-v}}$$

and extends to all of $\mathcal{T}_f$ by analytic continuation. Notice that σ_f actually takes $(-\infty, 0)$ to itself and has a unique attracting fixed point at $v \approx -1.3247$, the real negative root of $v - v^3 - 1$. Thus the unique rational map (up to holomorphic conjugacy) equivalent to f is $z \to 1.3247z^2 - 1.3247$. Note that this map is more usually conjugated to the 'airplane' $z \to z^2 - 1.7549$.

4.1. **Statement of the Theorem.**

Definition 4.4 (Orbifold)**.** Define $\nu : S^2 \to \mathbb{N} \cup \infty$ to be the smallest function for which $\nu(x) = 1$ when x is not in $P(f)$, and $\nu(x)$ is a multiple of $\nu(y)\deg_y(f)$ whenever $f(y) = x$. Then $O(f)$ is the orbifold with underlying space S^2 and a cone point of order $\nu(x)$ at each $x \in P(f)$.

Note that ν is finite if each critical point is strictly preperiodic (i.e. preperiodic but not periodic), and is infinite on each periodic orbit that contains a critical point. An orbifold has an Euler characteristic $\chi(O(f)) \in \mathbb{Q}$, where a cone point of order n counts as $1/n$ of a point. It is hyperbolic if $\chi < 0$.

Definition 4.5 (Multicurve)**.** A *multicurve* Γ is a finite disjoint union of non-parallel non-peripheral isotopy classes of simple closed curves in $S^2 - P(f)$. It is f*-stable* if for all $\gamma \in \Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is isotopic in $S^2 - P(f)$ into Γ.

If $\gamma_{i,j,\alpha}$ is a component of $f^{-1}(\gamma_j)$ isotopic to γ_i then f maps $\gamma_{i,j,\alpha}$ to γ_i with degree $d_{i,j,\alpha}$. Let $\mathbb{R}^\Gamma$ be the space of weights on Γ and let $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ be the matrix with coefficients $f_\Gamma(\gamma_j) = \sum_{i,\alpha} d_{i,j,\alpha}^{-1}\gamma_i$. Let $\lambda(\Gamma, f)$ denote the Perron-Frobenius eigenvalue of f_Γ.

With these definitions, Thurston's criterion may be stated as follows:

Theorem 4.6 (Thurston)**.** *A post-critically finite branched cover $f : S^2 \to S^2$ with hyperbolic orbifold is equivalent to a rational function if and only if for any f-stable multicurve Γ we have $\lambda(\Gamma, f) < 1$. Moreover, such a rational function is unique up to (holomorphic) conjugacy.*

An f-stable multicurve Γ with $\lambda \geq 1$ is called a (Thurston) *obstruction* and an f that admits one is said to be *obstructed*.

4.2. **Teichmüller Space.** The *Teichmüller Space* $\mathcal{T}_f$ is the space of isotopy classes rel. $P(f)$ of conformal structures on S^2. Formally:

Definition 4.7 (Teichmüller Space)**.** The Teichmüller space $\mathcal{T}_f$ is the space of equivalence classes of diffeomorphisms $\phi : (S^2, P(f)) \to \hat{\mathbb{C}}$ where $\phi_1 \sim \phi_2$ if there is a Möbius transformation $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for which ϕ_2 and $h\phi_1$ are isotopic rel. $P(f)$.

Of course, $\mathcal{T}_f$ is isomorphic to the Teichmüller space of the sphere with $|P(f)|$ marked points.

Now suppose we are given $\tau \in \mathcal{T}_f$ represented by $\phi : S^2 \to \hat{\mathbb{C}}$. The map f determines a branched cover of $\hat{\mathbb{C}}$; if we pull back the conformal structure on $\hat{\mathbb{C}}$ under this map and uniformize the result, we may realize this by a holomorphic branched cover $f_\tau : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and pulling back ϕ under f gives $\phi' : (S^2, P(f)) \to \hat{\mathbb{C}}$ for which $\phi f = f_\tau \phi'$. Define the *skinning map* $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ to be the map that takes the class of ϕ to the class of ϕ'.

Lemma 4.8. *The map f is equivalent to a rational map if and only if σ_f has a fixed point ϕ.*

Proof. That a rational map equivalent to f gives a fixed point follows essentially from definitions. Conversely if $\phi \sim \phi'$ with notation as above, there is a Möbius map h for which ϕ_2 and $h\phi_1$ are isotopic rel. $P(f)$, and then $f_\tau h$ is a rational map equivalent to f. □

Different marked conformal structures on a Riemann surface are related by quasiconformal homeomorphisms represented by Beltrami differentials. Thus, the tangent space to $\mathcal{T}_f$ at ϕ is represented by Beltrami fields $\mu d\bar{z}/dz \in B(\hat{\mathbb{C}})$ modulo those whose tangent vector field v preserves the (marked) conformal structure. This holds if and only if v vanishes on $P := \phi(P(f))$.

The dual of $B(\hat{\mathbb{C}})$ is the space of L^1 quadratic forms $q(z)dz^2$. For such a quadratic form, consider what it would mean for $\int q v_{\bar{z}} = 0$ for all v vanishing on P. Approximating q by convolution with a bump function and applying integration by parts, we obtain $q_{\bar{z}} = 0$ weakly on $\hat{\mathbb{C}} - P$; by Weyl's Lemma, q is holomorphic on $\hat{\mathbb{C}} - P$, and by a local calculation one finds that q can have at worst a simple pole at points of P. Thus:

Lemma 4.9. *The cotangent space to $\mathcal{T}_f$ at τ represented by ϕ is isomorphic to the space $Q(P)$ of holomorphic quadratic differentials on $\hat{\mathbb{C}} - P$ with at worst simple poles on P, where $P = \phi(P(f))$.*

Furthermore, we have

Lemma 4.10. *The dimension of $Q(P)$ is equal to $|P|-3$.*

In particular, $Q(P)=0$ unless $|P|\geq 4$.

Proof. This follows from Riemann-Roch, but it is elementary to show directly. WLOG we may assume ∞ is not in P. The ratio of any two quadratic meromorphic differentials on $\hat{\mathbb{C}}$ is meromorphic, and therefore rational. So $Q(P)$ consists precisely of functions of the form $(p(z)dz^2)/(\prod_{z_i\in P}(z-z_i))$ where $p(z)$ is a polynomial of degree at most $|P|-4$. $\square$

Beltrami differentials pull back under holomorphic maps. The adjoint of the pullback map is to push forward holomorphic quadratic differentials under *transfer*: if $F:(\hat{\mathbb{C}},P')\to(\hat{\mathbb{C}},P)$ is a holomorphic branched map and $q\in Q(P')$, then the value of F_*q at a point is the sum of the pullbacks of q under the (finitely many) local branches of F^{-1}. If q is integrable, so is F_*q, and if q has at worst simple poles at P', then F_*q has at worst simple poles on $F(P')\subset P$.

As above, let $\tau\in\mathfrak{T}_f$ be represented by ϕ, let ϕ' represent the class of $\sigma_f\tau$ and let $P':=\phi'(P(f))$. Then $f_\tau:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ takes P' into P. The derivative $d_\tau\sigma_f:T_\tau\mathfrak{T}_f\to T_{\sigma_f\tau}\mathfrak{T}_f$ has adjoint $(d_\tau\sigma_f)^*:Q(P')\to Q(P)$ and evidently $(d_\tau\sigma_f)^*=(f_\tau)_*$.

Let $F:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ be any rational map of degree d, and q a nonzero meromorphic quadratic differential with at worst simple poles at $Z\subset\hat{\mathbb{C}}$. Transfer F_*q cannot increase the mass of q, and can only reduce it unless the arguments of q at the various points of $F^{-1}(z)$ are equal for all z. In this case, the ratio of F^*F_*q and q is real; since it is meromorphic, it is constant, and (a posteriori) therefore equal to the degree d. Hence $\|F_*q\|=\|q\|$ implies that $(d^{-1})F^*F_*q=q$ and therefore $F^{-1}(F(Z))\subset Z\cup C(F)$.

Lemma 4.11. *If f has hyperbolic orbifold $O(f)$ then $d\sigma_f^2$ has norm strictly <1.*

Proof. If $\|(f_\tau)_*\|=1$ then there is nontrivial q with simple poles at $Z\subset P'$ for which $f_\tau^{-1}(f_\tau(Z))\subset P'\cup C(f_\tau)$. Since q is nontrivial, $|Z|\geq 4$. If we pull back $Z_f:=(\phi')^{-1}(Z)$ then $Z_f\subset P(f)$ and

$$\begin{aligned}f^{-1}(f(Z_f)) &= ((\phi')^{-1}f_\tau^{-1}\phi)(\phi^{-1}(f_\tau(Z)))\\ &= (\phi')^{-1}(f_\tau^{-1}(f_\tau(Z)))\subset(\phi')^{-1}(P'\cup C(f_\tau))=P(f)\cup C(f)\end{aligned}$$

In other words, there is a subset $W:=f(Z_f)\subset P(f)$ for which $f^{-1}(W)\subset P(f)\cup C(f)$.

For each vertex z of $P(f)\cup C(f)$ define the *indegree* $\iota(z)$ to be the sum of $\deg_f(w)$ over all $w\in f^{-1}(z)$ in $P(f)\cup C(f)$. Then $\sum_z\deg_f(z)=\sum_z\iota(z)$. Furthermore,

$$\sum_z\deg_f(z)\leq|P(f)|+|C(f)|+2d-2 \text{ and } \sum_z\iota(z)\geq|W|(d-1)+|P(f)|$$

It follows that

$$|W|(d-1)\leq|C(f)|+2d-2\leq 4(d-1)$$

and if equality holds, then every critical point is simple, $f(C(f))\subset W$, and $W\cap C(f)$ is empty.

In this last case, either $f(W)\subset W$ in which case $W=P(f)$ and $O(f)$ is a Euclidean orbifold with 4 cone points of order 2, or else $W':=f^{-1}(W)-C(f)$ does not satisfy $f^{-1}(W')\subset P(f)\cup C(f)$, in which case (repeating the argument above with $f_{\sigma_f\tau}$ in place of f_τ) we have $\|(f_{\sigma_f\tau})_*\|<1$. $\square$

This lemma shows if $O(f)$ is hyperbolic then a rational map equivalent to f is unique up to holomorphic conjugacy.

4.3. **Projection to moduli space.** The Teichmüller space $\mathcal{T}_f$ projects to the moduli space $\mathcal{M}_f$ of injections $i : P(f) \to \hat{\mathbb{C}}$ up to composition with a Möbius transformation. The skinning map σ_f does not descend to $\mathcal{M}_f$, but it *does* descend to a finite intermediate cover $\sigma_f : \tilde{\mathcal{M}}_f \to \mathcal{M}_f$.

Given $P := i(P(f))$ there are only finitely many degree d connected branched coverings $g : X \to \hat{\mathbb{C}}$ branched over P, up to isomorphism; they are determined by (transitive) conjugacy classes of actions of $\pi_1(\hat{\mathbb{C}} - P)$ on a d element set. For such a g with X homeomorphic to S^2, choose an injection $i' : P(f) \to g^{-1}(P) \subset X$ and consider the finite set of pairs (g, i') for which there are homeomorphisms $\phi : S^2 \to \hat{\mathbb{C}}$ and $\phi' : S^2 \to X$ with $\phi|P(f) = i$ and $\phi'|P(f) = i'$, and such that $\phi f = g\phi'$.

The set of pairs (g, i') is the fiber over i of a finite covering $\tilde{\mathcal{M}}_f \to \mathcal{M}_f$. If $\tau \in \mathcal{T}_f$ represented by ϕ maps to the class of i, then $\sigma_f\tau$ represented by ϕ' maps to the class of $\psi i' : P(f) \to \hat{\mathbb{C}}$ where $\psi : X \to \hat{\mathbb{C}}$ is a holomorphic isomorphism. Thus $\sigma_f : \tilde{\mathcal{M}}_f \to \mathcal{M}_f$ sending (g, i') to $\psi i'$ is the desired map.

Lemma 4.12. *Suppose $O(f)$ is hyperbolic, and $\tau_i := \sigma_f^i(\tau)$ projects to $\pi(\tau_i) \in \mathcal{M}_f$. Then τ_i converges in $\mathcal{T}_f$ if and only if $\pi(\tau_i)$ lie in a compact subset of $\mathcal{M}_f$.*

Proof. One direction is obvious, so suppose $\pi(\tau_i)$ lie in a compact subset of $\mathcal{M}_f$; or equivalently, their lifts $\tilde{\pi}(\tau_i) \in \tilde{\mathcal{M}}_f$ lie in a compact subset of $\tilde{\mathcal{M}}_f$.

Let δ_0 be a curve in $\mathcal{T}_f$ from τ_0 to τ_1 and let $\delta_i := \sigma_f^i(\delta_0)$. let $\tilde{\pi}(\delta_i)$ be the projection of δ_i to $\tilde{\mathcal{M}}_f$. The length of each δ_i is no more than that of δ_0, so because $\tilde{\pi}(\tau_i)$ lie in a compact region of $\tilde{\mathcal{M}}_f$, so do the $\tilde{\pi}(\delta_i)$.

But σ_f is uniformly strictly contracting on compact subsets of $\tilde{\mathcal{M}}_f$, and therefore the lengths of the δ_i form a geometric series and τ_i converge in $\mathcal{T}_f$. ∎

4.4. **Moduli of Annuli.** We are thus reduced to showing for f with hyperbolic orbifold $O(f)$, that the iterates of the skinning map $\sigma_f^i(\tau)$ project to a compact region in moduli space if and only if there is no f-stable multicurve with $\lambda(\Gamma, f) \ge 1$.

A sequence diverges in moduli space if and only if the associated Riemann surfaces develop thin 'necks' — conformally speaking, if there are essential embedded annuli $A \subset \hat{\mathbb{C}} - P$ whose moduli goes to infinity.

For a Euclidean annulus A with circumference c_A and height h_A the *modulus* $\mu(A)$ is h_A/c_A. Any annulus is conformally equivalent to a Euclidean annulus (double it and uniformize the result as a rectangular torus), and its modulus is the modulus of its Euclidean uniformizer. One may estimate the modulus in any conformal metric by the method of *extremal length*.

If X is any collection of rectifiable curves in a Riemann surface, and ρ is any metric in the conformal class, define $\ell_\rho(X)$ to be the minimal length of a curve in X and α_ρ the area of the surface. Then

$$E(X) := \sup_\rho \frac{\ell_\rho(X)^2}{\alpha_\rho}$$

is called the *extremal length* of X.

Lemma 4.13. *For an annulus A, if X denotes the collection of essential simple closed curves in A, the extremal length $E(X)$ is equal to $1/\mu(A)$.*

Proof. In the Euclidean metric, $\ell_\rho(X) = c_A$ and $\alpha_\rho = c_A h_A$ so $E(X) \geq c_A/h_A = 1/\mu(A)$. On the other hand, if we scale the metric by ρ then in Euclidean coordinates $x+iy$ on A, where $x \in [0, c_A]$ and $y \in [0, h_A]$, for any y

$$\ell_\rho(X) \leq \int_0^{c_A} \rho(x+iy)dx \text{ so that } \ell_\rho(X)h_A \leq \int_0^{h_A}\int_0^{c_A} \rho(x+iy)dxdy$$

But then by Cauchy–Schwarz,

$$\left(\int_0^{h_A}\int_0^{c_A} \rho(x+iy)dxdy\right)^2 \leq c_A h_A \int_0^{h_A}\int_0^{c_A} \rho(x+iy)^2 dxdy = c_A h_A \alpha_\rho$$

so $\ell_\rho(X)^2/\alpha_\rho \leq c_A/h_A = 1/\mu(A)$. □

Using this estimate we may now prove one direction of the desired result.

Lemma 4.14. *If f is a rational map then for every f-stable multicurve Γ one has $\lambda(\Gamma, f) \leq 1$. Furthermore, if $O(f)$ is hyperbolic then $\lambda(\Gamma, f) < 1$.*

Proof. Let Γ be an f-stable multicurve, and let v be a Perron-Frobenius eigenvector for f_Γ with eigenvalue λ. Then there exists a quadratic holomorphic differential $q \in Q(P)$ of norm 1 with annuli of closed trajectories $A_1, \cdots, A_n$ in the homotopy classes of $\gamma_1, \cdots, \gamma_n$ and with moduli h_i/c_i proportional to the coefficients of v. The differential q gives $\hat{\mathbb{C}}$ a branched Euclidean structure in which $|q|$ becomes the area form (thus $\hat{\mathbb{C}}$ has total area 1). In this metric, the closed trajectories winding around each A_j are the unique length-minimizing geodesics of length c_j (the circumference of A_j) in their isotopy class. Furthermore, each A_j has area $h_i c_i$, so $\sum_i h_i c_i = 1$.

For any other annulus $A \subset \hat{\mathbb{C}} - P$ homotopic to some A_j, if ℓ is the minimal q-length of an essential simple closed curve in A, we have $\ell \geq c_j$, the circumference of A_j. Thus by the method of extremal length, $\int_A |q| \geq c_j^2 \mu(A) = c_j^2 h_A/c_A$.

For each annulus A_j the preimage $f^{-1}(A_j)$ consists of a union of annuli $A_{i,j,\alpha}$ with core isotopic to A_i, and where $f : A_{i,j,\alpha} \to A_j$ has degree $d_{i,j,\alpha}$. Thus the modulus of $A_{i,j,\alpha}$ is $\mu(A_j)/d_{i,j,\alpha} = h_j/(c_j d_{i,j,\alpha})$.

But now we may estimate

$$1 = \int_{\hat{\mathbb{C}}} |q| = \sum_{i,j,\alpha} \int_{A_{i,j,\alpha}} |q| \geq \sum_{i,j,\alpha} c_i^2 \mu(A_{i,j,\alpha}) = \sum_{i,j,\alpha} c_i^2 \frac{h_j}{c_j d_{i,j,\alpha}} = \sum_i \lambda \frac{h_i}{c_i} c_i^2 = \lambda$$

Equality holds if and only if it holds for every application of the extremal length estimate. But this holds if and only if every $A_{i,j,\alpha}$ is a horizontal Euclidean annulus in the branched Euclidean structure, in which case $f^* q = \pm q$ which is impossible for $O(f)$ hyperbolic. □

4.5. **Geometry of thin tubes.** Comparisons between hyperbolic and conformal geometry of Riemann surfaces are difficult in general, but simplify asymptotically where surfaces are thin. Such regions correspond on the hyperbolic side to Margulis tubes around short geodesics, and on the conformal side to annuli of large moduli.

The following lemma gives an elementary relation between conformal modulus and hyperbolic length.

Lemma 4.15. *Let A be a Margulis tube around a geodesic of length ℓ. Then the modulus of A is $\pi/\ell - O(1)$.*

Proof. Let $R \subset \mathbb{C}$ be the Euclidean rectangle consisting of z with real part in $[0, \ell]$ and imaginary part in $[0, \pi]$. Let e^R in the upper half-plane be the image of R under exponentiation. If A' is the Euclidean annulus obtained from R by gluing the left and right sides by translation, then $\mu(A') = \pi/\ell$; this translation exponentiates to a hyperbolic isometry identifying edges of e^R whose quotient has a core geodesic of length ℓ. The Margulis tube A is obtained from A' by cutting off subannuli of the ends of (uniformly) bounded modulus. □

From this we may deduce the following estimate:

Lemma 4.16. *Let X be a complete hyperbolic surface, and $P \subset X$ a finite set. Let $X' = X - P$ and give X' its complete hyperbolic metric. Let γ be a simple closed geodesic on X and let γ_i be the (necessarily simple) closed geodesics on X' which are homotopic to γ in X and which have length in X' less than a Margulis constant. Let ℓ be the hyperbolic length of γ in X, and ℓ_i the hyperbolic lengths of γ_i in X'. Then*

$$\frac{1}{\ell} - \sum \frac{1}{\ell_i} = O(|P|)$$

Proof. Notice first that every $\ell_i > \ell$ since inclusion $X' \to X$ is necessarily 1-Lipschitz with respect to the hyperbolic metrics. So if ℓ is bigger than the Margulis constant, there are no γ_i.

Let $A \subset X$ be a Margulis tube for γ. The Margulis tubes $A_i \subset X'$ around the γ_i are disjoint, and inject into A. If we give A a flat Euclidean structure with circumference 1 and height π/ℓ then $A - \cup A_i$ is contained in a union of subannuli, each of bounded height, one around each point of $P \cap A$ and one around each cuff. Thus $\mu(A) - \sum \mu(A_i) = O(|P|)$ and the lemma is proved. □

4.6. **Conclusion of the argument.** Let $\tau \in \mathcal{T}_f$ and define $\tau_n = \sigma_f^n \tau$. We suppose that $O(f)$ is hyperbolic, so that τ_n converges to a fixed point unless the images in $\mathcal{M}_f$ are unbounded. The only way that the sequence of Riemann surfaces $(\hat{\mathbb{C}}, P_n)$ may diverge in $\mathcal{M}_f$ is if the hyperbolic structures on $X_n := \hat{\mathbb{C}} - P_n$ develop arbitrarily thin necks.

We have rational maps $f_{\tau_n} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Let $X'_n \subset X_n$ be $f_{\tau_n}^{-1}(X_{n-1})$, so that $f_{\tau_n} : X'_n \to X_{n-1}$ is a degree d cover. Thus, for every geodesic γ of X_{n-1} of hyperbolic length ℓ, the hyperbolic length of every preimage in X'_n has length at most $d\ell$, and the same is true of its image in X_n.

By definition of the Teichmüller distance, there are K-quasiconformal homeomorphisms $\psi_n : X_n \to X_{n-1}$, where $\log(K)$ is (twice) the Teichmüller distance from τ_n to τ_{n-1}, which may be bounded independent of n because σ_f is distance non-increasing.

A K-quasiconformal map distorts the moduli of annuli by at most a factor of K, so it stretches the lengths of short geodesics by at most a factor of K. Thus if ℓ is sufficiently small, and $\beta \subset X'_n$ is a curve of length $\le d\ell$, the hyperbolic length of the geodesic representative of $\psi_n(\beta)$ in X_{n-1} is at most $Kd\ell$. It follows that if $\Gamma \subset S^2 - P$ is a multicurve whose image in X_{n-1} consists of a maximal collection of geodesics of length less ϵ, then if ϵ is small enough and there are no geodesics with length in the interval $[\epsilon, Kd\epsilon]$, we may deduce that Γ is f-stable and conversely any sufficiently short geodesic on X'_n is in the preimage of some curve in Γ.

Now, if $\lambda < 1$, then because the cardinality of $|\Gamma|$ is bounded, there are only finitely many possible matrices f_Γ, and therefore after replacing f by an iterate f^m if necessary where m is *independent* of Γ, we may assume that f_Γ has operator norm at most $1/2$ (say).

Let γ_i be the curves of Γ, and let v_{n-1} be the vector with coefficients $1/\ell_i$ where ℓ_i is the length of the geodesic representative of γ_i in X_{n-1}. Pull back these geodesics under f_{τ_n} to geodesics on X'_n; this gives a family of geodesics $\gamma_{i,j,\alpha}$ with lengths $\ell_{i,j,\alpha} = d_{i,j,\alpha}\ell_i$.

If ℓ'_i is the length of the geodesic representative of γ_i in X_n, then by Lemma 4.16 we have

$$\sum_{i,\alpha} \frac{1}{\ell_{i,j,\alpha}} = \frac{1}{\ell'_j} + O(|C(f)|)$$

Thus if v_n is the vector with coefficients $1/\ell'_i$ we have $v_n = f_\Gamma v_{n-1} + o(|v_{n-1}|)$. In particular, if $|v_{n-1}|$ is sufficiently large, then $|v_n| \le (1/2)|v_{n-1}| + o(|v_{n-1}|) < |v_{n-1}|$. Thus, the $|v_n|$ are uniformly bounded, so that no geodesic in X_n can get too short, and therefore the projection of τ_n stays in a bounded region of moduli space. It follows that τ_n converge in Teichmüller space and Theorem 4.6 is proved.

4.7. **Topological polynomials.** Unfortunately the criterion in Theorem 4.6 is rather hard to check directly, since an obstructing multicurve might be rather complicated. For certain classes of branched maps there is more structure that can be exploited to simplify the search for such an obstruction.

A branched map $f : S^2 \to S^2$ is a *topological polynomial* if there is a critical point (which we may call ∞) that maps to itself with degree equal to $\deg(f)$. In other words, $f^{-1}(\infty) = \infty$. Note that this implies $\nu_f(\infty) = \infty$. If the postcritical orbit (not counting ∞) contains at least three points, O_f is hyperbolic.

The existence of a totally invariant point ∞ has the following corollary:

Lemma 4.17. *Let f be a topological polynomial. Then for every open disk $D \subset S^2 - \infty$, every component of $f^{-1}(D)$ is an open disk.*

Proof. Let $X := S^2 - \bar{D}$. Then f maps each component of $f^{-1}(X)$ properly onto X. But X contains ∞ which is totally invariant, so $f^{-1}(X)$ is connected. The lemma follows. □

Definition 4.18. Let $f : S^2 \to S^2$ be a post-critically finite branched map. A *Levy cycle* Λ is a cyclically ordered collection of essential non-boundary parallel curves $\gamma_1, \cdots, \gamma_k$ for which

(1) Λ is a subset of some f-stable multicurve Γ; and
(2) for each i the curve γ_i is isotopic rel. $P(f)$ to exactly one component γ' of $f^{-1}(\gamma_{i+1})$, and $f : \gamma' \to \gamma_{i+1}$ has degree 1.

Note that the existence of a Levy cycle implies $\lambda(\Gamma, f) \geq 1$ so a branched map with a Levy cycle is obstructed. Actually, it is easy to see that the existence of a Levy cycle is an obstruction, since γ' has the same length in the hyperbolic metric on $\hat{\mathbb{C}} - f^{-1}(P(f))$ as γ_{i+1} in $\hat{\mathbb{C}} - P(f)$, so that the geodesic representative of γ_i in $\hat{\mathbb{C}} - P(f)$ is strictly shorter than the geodesic representative of γ_{i+1} in $\hat{\mathbb{C}} - P(f)$, which (by induction on i) gives a contradiction.

Conversely, Bielefeld–Fisher–Hubbard [7] Thm. 5.5 show that an obstructed topological polynomial has a Levy cycle:

Theorem 4.19 (Obstructed polynomial has Levy cycle)**.** *If the topological polynomial f has an obstruction Γ then f has a Levy cycle $\Lambda \subset \Gamma$.*

Proof. We assume without loss of generality that Γ is minimal. This implies that every $\gamma \in \Gamma$ is isotopic rel. $P(f)$ to a component of $f^{-1}(\gamma')$ for some $\gamma' \in \Gamma$.

A curve $\gamma \in \Gamma$ is *negligible* if $\|f_\Gamma^n(\gamma)\| \to 0$ as $n \to \infty$, and *essential* otherwise. Since f_Γ has non-negative entries, if γ is negligible then so is every nonperipheral component of $f^{-1}(\gamma)$. Conversely, if γ is essential then some component of $f^{-1}(\gamma)$ is essential. Let $\Gamma_e \subset \Gamma$ denote the essential curves.

A component $\gamma \in \Gamma_e$ is *innermost* if the disk $D(\gamma)$ bounded by γ and disjoint from ∞ contains no curves isotopic to any $\gamma' \in \Gamma_e - \gamma$. Let $\Gamma_i \subset \Gamma_e$ denote the (nonempty) set of innermost curves. The key point is this: if $\gamma \in \Gamma_i$ then exactly one component of the isotopy class of $f^{-1}(\gamma)$ is essential, and this component is innermost.

To see why, let $\gamma' \in \Gamma_e$ be isotopic rel. $P(f)$ to a component of $f^{-1}(\gamma)$. The components of $f^{-1}(D(\gamma))$ are all disks by Lemma 4.17 and γ' bounds a disk U isotopic rel. $P(f)$ to a component of $f^{-1}(D(\gamma))$. Any essential $\beta \in \Gamma_e$ in U is isotopic rel. $P(f)$ to a preimage of some essential δ. But δ must lie outside $D(\gamma)$ since γ is innermost by hypothesis. Thus β is isotopic rel. $P(f)$ outside U which implies β is trivial rel. $P(f)$ which is absurd. This contradiction shows that every essential component of $f^{-1}(\gamma)$ is innermost.

Now suppose $\gamma_1, \gamma_2 \in \Gamma_i$ have preimages $\gamma_j' \subset f^{-1}(\gamma_j)$ that are both isotopic rel. $P(f)$ to some $\eta \in \Gamma_i$. If $X = P(f) \cap D(\eta)$ then $f(X)$ is contained in both $D(\gamma_1)$ and $D(\gamma_2)$. But these disks are disjoint because both γ_j are innermost.

It follows that distinct innermost curves in Γ_i have essential preimages that are distinct and innermost. Since there are only finitely many innermost curves, it follows that each $\gamma \in \Gamma_i$ has a *unique* preimage isotopic to some $\gamma' \in \Gamma_i$. Hence there is a cycle $\Gamma_L := \gamma_1, \cdots, \gamma_k$ of innermost essential curves so that $f^{-1}(\gamma_{i+1})$ contains exactly one component γ' isotopic to γ_i rel. $P(f)$. Since Γ_L is f_Γ-invariant modulo negligible curves, by minimality we have $\Gamma_L = \Gamma_i = \Gamma_e$.

It remains to show that $\gamma' \to \gamma_{i+1}$ has degree 1 for all i. But the Perron–Frobenius eigenvalue of f_Γ on the subspace spanned by Γ_L is equal to the reciprocal of the product of these degrees. Since the eigenvalues are strictly less than 1 on the negligible curves, and since (by definition) $\lambda \geq 1$, it follows that every degree is 1 so that Γ_L forms a Levy cycle as claimed. $\square$

Corollary 4.20. *Suppose that f is a topological polynomial. If every critical point is periodic (or more generally lands in a periodic cycle that contains a possibly different critial point) then f is not obstructed.*

Proof. For each γ_i in a Levy cycle isotopic rel. $P(f)$ to a component γ' of $f^{-1}(\gamma_{i+1})$ the disk $D(\gamma')$ maps to $D(\gamma_{i+1})$ with degree 1 and therefore $D(\gamma')$ (and, consequently, $D(\gamma_i)$) can contain no critical points. □

4.8. **Hubbard's Rabbit problem.** A quadratic polynomial $f_c : z \to z^2 + c$ has two critical points — ∞ (which is completely invariant) and 0. The critical point 0 is periodic with period dividing 3 if and only if $(c^2 + c)^2 + c = 0$ and has period exactly 3 if and only if $c^3 + 2c^2 + c + 1 = 0$. The three solutions give rise to Douady's 'rabbit' $f_R(z) :\approx z^2 + (-0.1226 + 0.7449i)$, the 'corabbit' $f_C(z) :\approx z^2 + (-0.1226 - 0.7449i)$ and the 'airplane' $f_A(z) :\approx z^2 - 1.7549$.

Let f_R be the rabbit polynomial, and let γ be a simple loop in $\mathbb{C}$ enclosing c and $c^2 + c$ (for $c \approx -0.1226 + 0.7449i$) but not 0. Let τ be a (right-handed) Dehn twist about γ. For each integer m the composition $\tau^m f_R$ is a post-critically finite branched covering of S^2 of degree 2 for which one critical point (∞) is fixed, and the other (0) is periodic of period 3.

By Corollary 4.20 these branched coverings are all unobstructed, and therefore each $\tau^m f_R$ is equivalent to exactly one of f_R, f_C, f_A. Hubbard's Rabbit problem asks: which one?

The answer was given by Bartholdi–Nekrashevych [5]:

Theorem 4.21 (Bartholdi–Nekrashevych)**.** *Let $m_i \in \{0, 1, 2, 3\}$ be the 4-adic digits of m (so that almost all $m_i = 3$ if m is negative). If one of the m_i is 1 or 2 then $\tau^m f_R$ is equivalent to f_A. Otherwise it is equivalent to f_R if $m \geq 0$ and to f_C if $m < 0$.*

5. The Mandelbrot Set

For $c \in \mathbb{C}$ the quadratic polynomial $f_c : z \to z^2 + c$ has a unique critical point at 0.

Definition 5.1. The *Mandelbrot set* $\mathcal{M} \subset \mathbb{C}$ is the set of $c \in \mathbb{C}$ for which 0 is not in the basin of attraction of ∞ for the map f_c.

Lemma 5.2. *The set $\mathcal{M}$ is compact and contained in the closed disk of radius 2.*

Proof. The property of 0 being in the basin of attraction of ∞ for f_c is evidently open in c, so $\mathcal{M}$ is closed.

If $|z| > |c| > 2$ then $|z^2 + c| - |z| > |z| - |c|$ and by induction $|f_c^n(0)|$ increases without bound so 0 is in the basin of ∞. □

Example 5.3. The point $-2 \in \mathcal{M}$. For, the orbit of 0 under f_{-2} is

$$0 \to -2 \to -2$$

Lemma 5.4. *If $c \in \mathcal{M}$ then $|f_c^n(0)| \leq 2$. Consequently $\hat{\mathbb{C}} - \mathcal{M}$ is connected.*

Proof. We have shown $|c| \leq 2$. So if $|f_c^n(0)| > 2$ for some n then as before $|f_c^n(0)|$ increases without bound and we would have $c \in \hat{\mathbb{C}} - \mathcal{M}$.

If $\hat{\mathbb{C}} - \mathcal{M}$ contained a bounded component U we would have $|f_c^n(0)| \to \infty$ in U but $|f_c^n(0)| \leq 2$ in ∂U which would violate the maximum principle. □

Lemma 5.5. *If $c \in \mathcal{M}$ then $J(f_c)$ is connected. Otherwise $J(f_c)$ is a Cantor set on which f_c is topologically conjugate to the 1-sided shift on a 2-letter alphabet.*

Proof. Let U be the basin of ∞, which is also the immediate basin because ∞ is completely invariant. If $c \in \mathcal{M}$ then the Böttcher map extends to a global conjugacy of f_c on U to $z \to z^2$ on $\mathbb{C} - \bar{\mathbb{D}}$. In particular, ∂U is connected. Since it is closed and f_c-invariant and contained in $J(f_c)$ it is equal to $J(f_c)$.

Otherwise let ϕ be a Böttcher map from a neighborhood of ∞ to a neighborhood of ∞ conjugating f_c to $z \to z^2$. Let E be a maximal open round disk neighborhood of ∞ on which ϕ^{-1} is defined. Then actually ϕ^{-1} extends continuously to ∂E, and $\phi^{-1}(\partial E)$ is a symmetric figure 8 in $\mathbb{C}$ whose double point is at 0. In other words, $D := \hat{\mathbb{C}} - \phi^{-1}(\bar{E})$ consists of two disjoint open disks $D = D_0 \cup D_1$. Since E is forward invariant under $z \to z^2$, it follows that $f_c^{-1}(D) \subset D$. By symmetry f_c^{-1} has two branches on each D_j, one whose image is contained in D_0 and one whose image is contained in D_1. By induction, $f_c^{-n}(D)$ has 2^{n+1} components that we may denote D_I for I a string of length $(n+1)$ in the alphabet $\{0, 1\}$, where D_{I0} and D_{I1} have closures contained in the interior of D_I. Each branch of f_c^{-1} is strictly contracting in the hyperbolic metric on D by the Schwarz Lemma, and therefore uniformly contracting on compact subsets (such as the closure of $f_c^{-1}(D)$). Thus the hyperbolic diameters of the components D_I go to zero geometrically as a function of $|I|$ and the intersection $\cap_I \cup_{|I|=n} \bar{D}_I$ is an f_c-invariant Cantor set on which f_c is conjugate to the 1-sided shift.

This Cantor set is precisely equal to $J(f_c)$. One way to see this is to observe firstly that its complement is contained in the basin of infinity, and secondly that it contains a dense subset of repelling periodic orbits. □

5.1. **Hyperbolic Components.** If $J(f_c)$ is connected, every component of $F(f_c)$ is a disk. If $F(f_c)$ contains an attracting cycle (other than the basin of infinity) the immediate basin of the attracting orbit must contain 0, the unique critical point. Thus such an attracting cycle, if it exists, is unique.

Lemma 5.6. *Let $a \in \mathcal{M}$ be a value for which f_a has an attracting periodic cycle of period m. Then a is in the interior of $\mathcal{M}$, and if $W \subset \mathcal{M}$ is the component of the interior of $\mathcal{M}$ containing a, then for every $c \in W$ there is an attracting periodic cycle $z_1(c), \cdots, z_m(c)$ for f_c where the $z_j(c)$ depend analytically on c.*

Proof. Attracting periodic orbits are stable, so there is certainly an open neighborhood U of a consisting of c for which f_c have an attracting periodic cycle of period m that varies analytically with c. Of course $U \subset \mathcal{M}$ so it is contained in some component W of the interior of $\mathcal{M}$. Now for each j let $g_j(c) := f_c^{jm}(0)$ be a function of $c \in W$. Since $c \in \mathcal{M}$ we have $|g_j(c)| \le 2$ so $\mathcal{G} := \{g_j\}$ is normal in W. By definition, $g_j(a) \to z(a)$, a periodic attracting point for f_a, and on U we have $g_j(c) \to z(c)$, a periodic attracting point for f_c. By normality, some subsequence of g_j converges on all W to g where $g(c) = z(c)$ on U.

Now let $V_m \subset \mathbb{C}^2$ be the variety of pairs (c, z) for which z is a fixed point of f_c^m. We have shown $(c, g(c)) \in V_m$ for $c \in U$ and since g is analytic, $(c, g(c)) \in V_m$ on all of W. Furthermore, the multiplier μ of f_c^m at $g(c)$ is evidently analytic in W and $|\mu| < 1$ in U. If $|\mu| \ge 1$ anywhere in W then we would have $|\mu| > 1$ on some open subset of W. But if $|\mu| > 1$ at c then $g(c)$ is a repelling periodic point for f_c. So if $f_c^{mj}(0) \to g(c)$ we must have $f_c^{mj}(0) = g(c)$ for some j. There are only countably many c with this property, so only countably many $c \in W$ where $|\mu| > 1$, thus actually $|\mu| < 1$ in W so in fact $z(c) := g(c)$

is an attracting periodic point of f_c of period dividing m for all $c \in W$. If the period were $< m$ for any c in W then reasoning exactly as above it would be $< m$ for all c in W hence also for a, contrary to assumption. □

Definition 5.7. A component W of the interior of $\mathcal{M}$ is a *hyperbolic component* if for some (and hence every) $a \in W$ the map f_a has an (analytically varying, unique) attracting periodic cycle of some fixed period m.

Theorem 5.8. *If W is a hyperbolic component, then ∂W is piecewise real analytic. Furthermore, if $\mu(c)$ is the multiplier of f_c^m at its (unique) attracting periodic orbit of period m, the map $\mu : W \to \mathbb{D}$ is a holomorphic isomorphism which extends continuously to a homeomorphism $\overline{W} \to \overline{\mathbb{D}}$.*

Proof. The multiplier μ is an analytic function on the variety $V_m \subset \mathbb{C}^2$ so the subset where $|\mu| < 1$ consists of a finite union of bounded open regions bounded by piecewise real analytic arcs. One such region $\tilde{W}$ projects to W, so ∂W is piecwise real analytic. Since W is a component of the interior of $\mathcal{M}$, the closure of the exterior $\hat{\mathbb{C}} - \mathcal{M}$ contains ∂W. Thus ∂W is a Jordan curve, and $\overline{W}$ is a topological disk. It remains to prove that $\mu : W \to \mathbb{D}$ is an isomorphism.

Let $U(c)$ be the component of $F(f_c)$ containing $z(c)$. Thus f_c^m maps $U(c)$ to itself with degree 2. Because $J(f_c)$ is connected, $U(c)$ is a disk, so $f_c^m|U(c)$ is conjugate to a unique Blaschke product of the form $z \to z(z+a)/(1+\bar{a}z)$ where $a = \mu \in \mathbb{D}$ is the multiplier. Conversely, for any $a \in \mathbb{D}$ as in Example 3.10 we can perform quasiconformal surgery on f_c to obtain a new quadratic map with an attracting periodic orbit of period m with multiplier a. This operation is analytic in a, and inverts the map $\mu : W \to \mathbb{D}$ which is therefore an isomorphism. □

It follows that each hyperbolic component contains a unique c (namely $\mu^{-1}(0)$) for which f_c has a superattracting cycle. In other words: there is a bijection between the hyperbolic components of $\mathcal{M}$ and the set of c for which 0 is periodic for f_c.

5.2. **The Mandelbrot Set is connected.**

Theorem 5.9 (Böttcher)**.** *Let z be a superattracting fixed point for f and thus also a critical point of some degree $(m-1)$. Then there is a neighborhood of z on which f is analytically conjugate to $z \to z^m$. Furthermore the conjugating map is unique up to multiplication by an $(m-1)$st root of unity.*

Proof. Böttcher's proof proceeds by writing down a power series for the conjugating map and showing that it converges on some neighborhood of the fixed point. This proof may be found in Milnor's book [13]. We give another proof using uniformization.

After conjugation by a Möbius transformation we may assume that $z = 0$ and f is of the form $z \to z^m + O(z^{m+1})$. Thus there is a small disk D about 0 with boundary α so that $f^{-1}(\alpha)$ contains a curve β enclosing a disk E with D contained in the interior of E, and $f : \beta \to \alpha$ an m-fold covering. Let A be the annulus $E - \mathrm{int}(D)$ and let Ω be obtained from E by gluing on countably many copies $A_n, n \in \mathbb{N}$ of A, where the inner boundary of A_n is glued to the outer boundary of A_{n-1} by $f : \beta \to \alpha$. Then Ω is isomorphic to $\mathbb{D}$, and f extends to $f : \Omega \to \Omega$ conjugate to a proper degree m map $g : \mathbb{D} \to \mathbb{D}$ with a single critical point of degree $(m-1)$ (which may be put at 0). Reflecting g along $\partial\mathbb{D}$ shows $g(z) = z^m$. $\square$

If f is a polynomial (of degree $d > 1$), ∞ is always a superattracting fixed point, and it is more natural to choose a Böttcher map ϕ that fixes ∞ and conjugates f near ∞ to $z \to z^d$ near ∞. If f is a depressed monic polynomial — i.e. of the form $f(z) = z^d + a_2 z^{d-2} + \cdots + a_d$ — then ϕ may be chosen uniquely to be tangent to the identity at ∞ to (at least) second order.

Let $c \in \mathbb{C} - \mathcal{M}$ and let ϕ conjugate $f_c : z \to z^2 + c$ near ∞ to $z \to z^2$. We may analytically continue ϕ^{-1} along radial lines from infinity all the way to the critical point, so certainly to the critical value c. Let $\alpha = \phi(c)$ and let $\Phi : \mathbb{C} - \mathcal{M} \to \mathbb{C} - \overline{\mathbb{D}}$ be the map that takes c to α. Douady–Hubbard showed:

Theorem 5.10 (The Mandelbrot Set is connected)**.** *The Böttcher map $\Phi : \mathbb{C} - \mathcal{M} \to \mathbb{C} - \overline{\mathbb{D}}$ is an isomorphism. Thus $\mathcal{M}$ is connected.*

Proof. It suffices to construct an inverse $\Psi : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - \mathcal{M}$. Given $v \in \mathbb{C} - \overline{\mathbb{D}}$ we want to find c so that the Böttcher map ϕ for $f_c : z \to z^2 + c$ takes c to v. The dynamics of $z \to z^2$ on $\mathbb{C} - \overline{\mathbb{D}}$ are conjugate to those of the unknown f_c near infinity, and we can pull back the conjugacy under iteration until the critical point: the point v has two preimages under $z \to z^2$ whereas c has only one preimage (i.e. 0) under f_c.

We will build a new Riemann surface Ω out of $\mathbb{C} - \overline{\mathbb{D}}$ by cut-and-paste so that $z \to z^2$ descends to a degree 2 proper map $F : \Omega \to \Omega$, and so that $v \in \Omega$ will have only one (critical) preimage. Here is how to do this. Let u_0, u_1 be the two preimages of v under $z \to z^2$ (i.e. $u_0, u_1 = \pm\sqrt{v}$). Let σ_0, σ_1 be the two radial segments in $\mathbb{C} - \mathbb{D}$ from the unit circle to u_0, u_1 respectively. If we cut along the σ_j and reglue edges in pairs (so that the left side of σ_0 glues to the right side of σ_1 and vice versa) we will obtain a new Riemann surface Ω_1 from $\mathbb{C} - \overline{\mathbb{D}}$, topologically a pair of pants, for which v 'has exactly one preimage' under $z \to z^2$. The two points u_0, u_1 in $\mathbb{C} - \overline{\mathbb{D}}$ are identified to a single point p in Ω_1.

However, the map $z \to z^2$ does not yet descend to Ω_1 and we must perform more modification. Let $\sigma_{00}, \cdots, \sigma_{11}$ be the four preimages of the two segments σ_0, σ_1 under $z \to z^2$. Cut these four segments and reglue (as before) so that the sides of each preimage of σ_0 are glued to the sides of some preimage of σ_1. There is a unique way to choose which preimages to match up so that the result of the gluing is a planar surface Ω_2, topologically a disk with four holes. Iterating this procedure we get a sequence of Riemann surfaces Ω_n, topologically a disk with 2^n holes, so that Ω_n is obtained from Ω_{n-1} by cut-and-paste. The limit Ω is a disk minus a Cantor set, and $z \to z^2$ descends to the desired map $F : \Omega \to \Omega$ with a unique critical point at p.

Any planar Riemann surface Ω is conformally isomorphic to a subset of $\hat{\mathbb{C}}$. By estimating the modulus of annuli converging to the ends of Ω one sees that $\hat{\mathbb{C}} - \Omega$ is the union of an isolated point (which we may take to be ∞) and a Cantor set K of Hausdorff dimension strictly less than 2. The map F on Ω extends continuously to $\hat{\mathbb{C}}$ and since it is conformal on a subset of full measure (and quasiconformal everywhere) it is actually conformal, hence a degree 2 polynomial. This is the desired inverse. $\square$

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University of Chicago, Chicago, Ill 60637 USA
E-mail address: dannyc@uchicago.edu