ALGEBRAIC TOPOLOGY, FALL 2016, MIDTERM

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This midterm exam was posted online on Friday, October 28, and is due before class Friday, November 4. Collaboration is not allowed, nor is the use of outside materials and textbooks. Hatcher and your class notes may be used to remember definitions, but not to copy calculations or proofs.

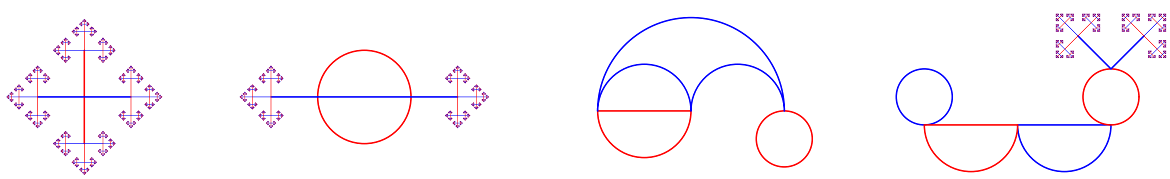
Problem 1. Give a CW complex structure and a Δ-complex structure on the (real) projective plane $\mathbb{R}P^2$. Use these structures to compute

(1) the fundamental group;
(2) the simplicial homology; and
(3) the cellular homology

Describe the isomorphism between cellular and simplicial homology, and the abelianization map from the fundamental group to H_1 (in terms of the presentations you gave above).

Problem 2. Show that the real projective plane is not a nontrivial cover of any other space.

Problem 3. Let X denote a wedge of two circles. What is $\pi_1(X)$? For each of the following covering spaces of X, identify the corresponding subgroup of $\pi_1(X)$, and say whether the cover is regular (i.e. normal) or not.

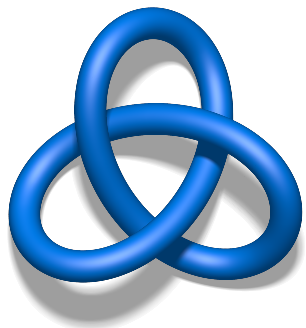


Problem 4. For each positive integer n let Σ_n denote the closed, oriented surface of genus n.

(1) Show that there is a degree p cover $\Sigma_m \to \Sigma_n$ if and only if $m = pn - p + 1$.
(2) If $m = pn - p + 1$ show by an explicit construction that there is a regular cover $\Sigma_m \to \Sigma_n$ with deck group $\mathbb{Z}/p\mathbb{Z}$. Is there more than one?
(3) In the case $n = 2$ and $p = 4$ (so that $pn - p + 1 = 5$) compute the action of the deck group on the homology of Σ_5 for the explicit convering you have constructed in part (ii).

Problem 5. Give an example of a pair of spaces X, Y with isomorphic fundamental groups, and with homeomorphic universal covers, for which the homology groups of X and Y are different.

Problem 6. The Trefoil knot is the embedded circle in the 3-sphere indicated by the figure below. Compute (e.g. with Seifert van-Kampen) the fundamental group and the homology of the complement of the knot in S^3.



Problem 7. State and prove the Five Lemma.

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