ALGEBRAIC TOPOLOGY, FALL 2013, FINAL

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This final exam was posted online on Friday, December 6, and is due by 11:30 on Friday, December 13. Collaboration is not allowed, nor is the use of outside materials and textbooks. Hatcher and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Bonus problems are just for fun, and do not contribute to the grade for the class.

Problem 1. Compute the cup product structure on the cohomology of a Klein bottle, for both integer and $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Problem 2. Show that $\pi_2(S^2 \vee S^1)$ is infinitely generated (hint: think about covering spaces).

Problem 3. Let M be a closed 3-manifold with $\pi_2(M) = 0$. Show that the fundamental group of M cannot contain a subgroup isomorphic to $\mathbb{Z}^4$, and if it contains a subgroup isomorphic to $\mathbb{Z}^3$, that subgroup is finite index.

Problem 4. Show that for any n, the covering projection $S^{2n} \to \mathbb{RP}^{2n}$ induces 0 in (integral) homology or cohomology (except in dimension 0) but this map is not null-homotopic.

Problem 5. Is there a map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ of negative degree (i.e. taking the generator of $H_4(\mathbb{CP}^2)$ to a negative multiple of itself)?

Problem 6. Give an example of two closed, connected manifolds whose homotopy groups are isomorphic in every dimension, but which are not homotopy equivalent.

Problem 7. A closed orientable n-manifold M is *spherical* if there is some map $f : S^n \to M$ of nonzero degree (i.e. for which the image of the generator of $H_n(S^n)$ is equal to a nonzero multiple of the generator of $H_n(M)$). Prove that if M is spherical and $n > 1$, then $\pi_1(M)$ is finite.

Problem 8. Let M be a closed, simply-connected 4-manifold. Let W be obtained from M by removing a point. Show that W is homotopy equivalent to a wedge of S^2s. (Bonus question: show that M can be obtained from a wedge of 2-spheres up to homotopy by attaching a 4-cell, and describe how the element of $\pi_3(\vee_i S_i^2)$ given by the attaching map determines the cup product pairing on $H^2(M;\mathbb{Z})$)

Bonus Problem 9. Show that $\mathbb{CP}^2$ is not the boundary of a (smooth) compact 5-manifold.

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