HOMEWORK 2 — EUCLIDEAN, HYPERBOLIC AND CONFORMAL GEOMETRY

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Problem 1. A subset $C \subset \mathbb{E}^3$ is *convex* iff for each pair of distinct points $x, y \in C$, the line segment joining x to y is contained in C. A *convex combination* of points $v_1 \dots v_m$ of $\mathbb{E}^n$ is a linear combination of the form $t_1 v_1 + \cdots + t_m v_m$ such that $t_1 + \cdots + t_m = 1$ and $t_i \geq 0$ for all i.

(1) Show that a subset C of $\mathbb{E}^n$ is convex iff C contains every convex combination of points in C.
(2) The *convex hull* of a subset $S \subset \mathbb{E}^n$ is the intersection $C(S)$ of all the convex subsets of $\mathbb{E}^n$ containing S. Prove that $C(S)$ is the set of all convex combinations of points of S.
(3) Let G be the group of homeomorphisms (i.e. 1–1, continuous with a continuous inverse) of $\mathbb{E}^n$ which take convex sets to convex sets. Show that G is exactly the group of *affine transformations*: that is, the group of transformations of the form

$$x \to Ax + t$$

where $A \in GL(3, \mathbb{R})$ and t is a translation.

Problem 2. Let $S \subset \mathbb{E}^3$ be a sphere, and π a plane tangent to S at x. *Stereographic projection* from S to π is the map taking each point $p \in S$ to the intersection q of the line $x'p$ with π, where x' is the point on S opposite x. Show that stereographic projection can be extended to an inversion.

Problem 3. Let A, B, C be mutually tangent spheres. Let X_0 be a fourth sphere tangent to A, B and C. Construct a chain of spheres, beginning with X_0, where X_i is tangent to A, B, C and to X_{i-1}. Show that the chain closes up on the sixth sphere. Note: assume that the spheres are chosen so that their interiors are all disjoint.

Problem 4. Show that for $x, y \in \mathbb{D}^3$, the identity

$$\cosh(d_{\text{hyp}}(x, y)) = 1 + \frac{2|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}$$

holds, where d_{hyp} denotes the distance between points with the conformally invariant hyperbolic metric on $\mathbb{D}^3$, and $|x - y|$ denotes the Euclidean distance between x and y, and $|x|$ denotes the Euclidean distance from 0 to x.

Hint: show that the expression on the right hand side is invariant under a conformal transformation of $\mathbb{D}^3$.

Problem 5. Suppose S, T are two distinct spheres perpendicular to the unit sphere. Let i_S and i_T be the isometries of $\mathbb{D}^3$ in the hyperbolic metric which are defined by inversion in S and T respectively. Show that the fixed point set in $\mathbb{D}^3$ of the composition $i_S i_T$ is exactly the intersection $S \cap T$.

Problem 6. A *hyperbolic sphere* is the set of points in hyperbolic space equidistant from some point, the center. Show that the hyperbolic spheres in the conformal model $\mathbb{D}^3$ are exactly the Euclidean spheres completely contained in $\mathbb{D}^3$.

Problem 7. Recall that a *vector field* w on $U \subset \mathbb{R}^n$ is a smoothly varying choice of tangent vector $w(p) \in T_p\mathbb{R}^n$ for each $p \in \mathbb{R}^n$. A vector field is *conformal* if it is tangent to a 1–parameter family of conformal transformations. That is, if there is a family of conformal maps $\phi_t : U \to \mathbb{R}^n$ such that each ϕ_t is conformal, and if for each p, the tangent vector

$$\left.\frac{d\phi_t(p)}{dt}\right|_{t=0} = w(p)$$

Show that a vector field w on $\mathbb{R}^n$ is conformal iff

$$\frac{\partial w_i}{\partial x_j} = -\frac{\partial w_j}{\partial x_i}$$

for $i \neq j$, and

$$\frac{\partial w_i}{\partial x_i} = \frac{1}{n}\sum_k \frac{\partial w_k}{\partial x_k}$$

Here w_i denotes the ith component of the vector w_i, which can be thought of as a function on U. Deduce that for a conformal vector field w,

$$(n-1)\frac{\partial^2 w_i}{\partial x_i^2} + \sum_{i \neq j} \frac{\partial^2 w_i}{\partial x_j^2} = 0$$

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