HOMEWORK 3 — SPHERICAL GEOMETRY

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This homework is due October 20th at the start of class. Recall that $O(3,\mathbb{R})$ denotes the group of 3×3 matrices with real entries satisfying $A^tA = \text{id}$, and $SO(3,\mathbb{R})$ denotes the subgroup with determinant 1.

Problem 1. For vectors x, y, z, w in 3–space, prove the identity

$$(x\times y)\cdot(z\times w) = \begin{vmatrix} x\cdot z & x\cdot w \\ y\cdot z & y\cdot w \end{vmatrix}$$

Problem 2. Prove that two spherical triangles are congruent if and only if they have the same angles.

Problem 3. Let D be a regular dodecahedron. Calculate the dihedral angles of D — that is, the angles between the faces.

Hint: think about the projection of D from its circumcenter to its circumscribing sphere.

Problem 4. Let $A \in SO(3,\mathbb{R})$. Show that A has a real eigenvalue equal to 1. Show A is conjugate to a matrix of the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$.

Problem 5. Let $\mathbb{S}^3$ be the group of length 1 quaternions; that is, elements of the form $a + bi + cj + dk$ where multiplication of quaternions is determined by

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$$

and the length of the quaternion $a + bi + cj + dk$ is $\sqrt{a^2 + b^2 + c^2 + d^2}$.

Check first that this is a group, and that the elements of the form $a + bi$ form a subgroup isomorphic to S^1 (the group of length 1 complex numbers).

Show that there is an isomorphism $\mathbb{S}^3 \cong SU(2)$ where $SU(2)$ is the group of 2×2 complex matrices A with determinant 1 such that $\overline{A}^t A = \text{id}$. (Here $\overline{A}$ denotes the matrix whose entries are the complex conjugates of the entries of A)

Hint: think of $\mathbb{R}^4$ alternately as the quaternions, or as $\mathbb{C}^2$.

Problem 6. Define an equivalence relation on points (z, w) in S^3 by $(z_1, w_1) \sim (z_2, w_2)$ if and only if there is a nonzero complex number λ with $|\lambda| = 1$ such that $(\lambda z_1, \lambda w_1) = (z_2, w_2)$. Show that the equivalence classes of this equivalence relation are exactly the cosets of some S^1 in $SU(2)$. What is the coset space?

Problem 7. Let E be an invertible 3×3 real matrix, and let $J = E^tE$. Let $O(J,\mathbb{R})$ be the group of 3×3 real matrices A such that $A^tJA = J$. Show that $O(J,\mathbb{R}) \cong O(3,\mathbb{R})$, and find an explicit isomorphism.

Problem 8. Using the spherical law of cosines and the area formula, calculate the area and the shape of the largest spherical quadrilateral with perimeter equal to some fixed value t. Do the same for a spherical pentagon with perimeter equal to some fixed t. Assume t is reasonably small.

Conversely, determine the shape of a spherical quadrilateral or pentagon with fixed area of smallest perimeter.

Problem 9 (Foucault's pendulum — hard). A *pendulum* is a weight on the end of a string whose other end is fastened rigidly. Imagine a person walking around on a planet which is not rotating, for example on Earth's moon, carrying a pendulum. If the person stands still, the pendulum swings back and forth along a fixed axis. If the person walks in a straight line on the surface of the planet, the pendulum swings back and forth along the same apparent axis — that is, the axis makes a constant angle with the direction the person is facing. Show that if a person walks around a (polygonal) loop on the surface of the planet enclosing an area αr^2 where r is the radius of the planet, then the axis of the pendulum at the end of the loop has rotated from the axis of the pendulum at the start of the loop by an angle α.

Now imagine the person on Earth, holding the pendulum. Now the person stands still and the Earth rotates. Show that the axis of the pendulum appears to rotate, relative to the stationary observer. In 24 hours, the pendulum rotates through an angle $2\pi \sin(\beta)$ where β is the latitute.

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