HOMEWORK 2 — THE EUCLIDEAN PLANE

DANNY CALEGARI

This homework is due October 6th at the start of class. Recall that Aff$(\mathbb{E}^2)$ denotes the group of *affine transformations* — i.e. transformations preserving straight lines and incidence properties — of $\mathbb{E}^2$, Sym$(\mathbb{E}^2)$ denotes the group of *similarities* — i.e. transformations preserving angles and ratios of lengths — and Isom$(\mathbb{E}^2)$ denotes the group of *isometries* — i.e. transformations preserving angles and lengths. For a group $X(\mathbb{E}^2)$, the subgroup of orientation–preserving elements is denoted $X^+(\mathbb{E}^2)$.

Problem 1. Let H denote the subspace of $\mathbb{E}^2$ consisting of pairs of points (x, y) with $y \geq 0$. Let G be the subgroup of Sym$^+(\mathbb{E}^2)$ which stabilizes H; that is, which takes $H \to H$ in a 1–1 and invertible fashion. Show that G is isomorphic to Aff$^+(\mathbb{E}^1)$. In this way identify the "space of (oriented) lines" in $\mathbb{E}^2$ with the coset space Sym$^+(\mathbb{E}^2)/$Aff$^+(\mathbb{E}^1)$.

Problem 2. Using the fact that G is a group, prove the following geometric theorem: let C_1, C_2, C_3 be three circles of different radii in $\mathbb{E}^2$ with disjoint interiors, and for each pair C_i, C_j let l_{ij}^k with $k = 1, 2, 3, 4$ be the 4 lines which are tangent to both C_i and C_j. These intersect in 6 points, but only 2 of these points are on the line joining the centers of C_i and C_j. Call these two special points p_{ij}^1 and p_{ij}^2. In this way we get 6 points, $p_{12}^1, p_{12}^2, p_{13}^1, p_{13}^2, p_{23}^1, p_{23}^2$. Call this collection of points P. Show that there are 4 special lines, each of which intersects P in 3 points.

Problem 3. Identifying the group of translations with $\mathbb{C}, +$ and the group of similarities fixing a point p with $\mathbb{C}^*, \times$, show that exponentiation (i.e. $z \to e^z$) defines a homomorphism $\mathbb{C}, + \to \mathbb{C}^*, \times$. What is the kernel of this homomorphism? Show that it is onto. Use this to show that there are subgroups of $\mathbb{C}^*, \times$ which are isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$. What does the orbit of a point $q \in \mathbb{E}^2$ (with $p \neq q$) look like under the action of such a $\mathbb{Z} \times \mathbb{Z}$ in $\mathbb{C}^*$?

Problem 4. Fix some point $p \in \mathbb{E}^2$. Show that every element of Isom$^+(\mathbb{E}^2)$ can be written as a product $t_{x_0,y_0} r_\theta$ where r_θ denotes the rotation through angle θ about the point $p \in \mathbb{E}^2$, and t_{x_0,y_0} denotes translation through the vector (x_0, y_0).

Think of $\mathbb{E}^2$ as the plane $z = 1$ in $\mathbb{R}^3$. Using this identification, show that there is an isomorphism between Isom$^+(\mathbb{E}^2)$ and the group of 3×3 matrices of the form

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) & x_0 \\ -\sin(\theta) & \cos(\theta) & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 5. Show that there is a short exact sequence (that is, the image of one homomorphism is the kernel of the next)

$$1 \xrightarrow{i} \mathbb{R}^2 \xrightarrow{j} \text{Aff}(\mathbb{E}^2) \xrightarrow{\phi} GL(2, \mathbb{R}) \xrightarrow{\psi} 1$$

where the homomorphisms i, j are inclusions, and ϕ, ψ are quotients.

Show that this short exact sequence *splits*; that is, there is a homomorphism

$$s : GL(2, \mathbb{R}) \to \text{Aff}(\mathbb{E}^2)$$

such that $\phi \circ s = \mathrm{id}$, and therefore that $\mathrm{Aff}(\mathbb{E}^2)$ is a semi–direct product of $GL(2, \mathbb{R})$ and $\mathbb{R}^2$. What is the induced homomorphism $GL(2, \mathbb{R}) \to \mathrm{Aut}(\mathbb{R}^2)$?

Problem 6. Show that every translation can be written as a product of 2 reflections. Show that every rotation can be written as a product of 2 reflections.

Problem 7. Show that any two nonzero translations are conjugate in $\mathrm{Aff}(\mathbb{E}^2)$. Think of $\mathrm{Sym}(\mathbb{E}^2)$ as a subgroup of $\mathrm{Aff}(\mathbb{E}^2)$, and therefore think of the rotations in $\mathrm{Sym}(\mathbb{E}^2)$ as a particular subset of elements of $\mathrm{Aff}(\mathbb{E}^2)$. When are two rotations conjugate in $\mathrm{Aff}(\mathbb{E}^2)$? (Hint: the trace of a 2×2 matrix is invariant under conjugation by an element of $GL(2, \mathbb{R})$.)

Problem 8 (Hjelmslev's theorem). A *glide reflection* is the composition of two isometries: a reflection in a line l and a translation in a direction parallel with the line l. l is said to be the *axis* of the glide reflection.

Suppose $i \in \mathrm{Isom}(\mathbb{E}^2)$ is orientation–reversing. Show that i is either a reflection or a glide reflection. For any point x, show that the midpoint of $xi(x)$ lies on the "axis" of i (either the axis of reflection or the axis of the glide reflection). Let pq and $p'q'$ be two line segments of equal length. Show there is a unique orientation–reversing transformation of $\mathbb{E}^2$ to itself taking $p \to p'$ and $q \to q'$. Deduce that for any pair of line segments $pq, p'q'$ of equal length, and points $r \in pq, r' \in p'q'$ with $|pr| = |p'r'|$ and $|rq| = |r'q'|$, the midpoints of the segments pp', qq', rr' are collinear (they might be coincident).

Problem 9 (Hard). The function $f_n : z \to (1 + \frac{z}{n})^n$ is a function from $\mathbb{C}$ to itself which is generically $n \to 1$. Let S^1_n denote the circle of radius n in $\mathbb{C}$ with center $-n$. Let α_n be the rotation with center $-n$ through an angle of $2\pi/n$. Identifying $\mathbb{C}$ with $\mathbb{E}^2$, show that the sequence α_n of isometries *has a well–defined limit*. That is, show for each point p that the sequence of points $\lim_{n \to \infty} \alpha_n(p)$ limits to some point p_∞. Moreover, show that the transformation $p \to p_\infty$ is an isometry α_∞ of $\mathbb{E}^2$. What is α_∞? What does this have to do with the convergence

$$\lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n \to e^z$$

DEPARTMENT OF MATHEMATICS, HARVARD, CAMBRIDGE, MA 02138
E-mail address: dannyc@math.harvard.edu