138 FINAL EXAM

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This final exam was made available at noon on Thursday, January 4th. It is due by 3pm in my mailbox (or emailed to me) on Tuesday, January 9th. Only texts listed on the course home page and notes taken in class are permitted for consultation. Collaboration is not permitted until after the exam. **Answer only four of the first five problems and one of the last two problems.** Each problem is worth the same amount. This exam is worth 50% of the final grade.

Answer only four of the following five problems

Problem 1. Let G be a finitely generated group and let $\phi : G \to G$ be a isomorphism. Two elements α, α' are said to be *ϕ–conjugate* if there exists $\gamma \in G$ with

$$\alpha' = \gamma\alpha\phi(\gamma)^{-1}$$

Let $M(\phi)$ be the group obtained from G by adding a new generator z and adding the relations $zgz^{-1} = \phi(g)$ for all $g \in G$.

1. Show that $M(\phi)$ is a semi–direct product of G with $\mathbb{Z}$.
2. Show that any $x \in G$ is ϕ–conjugate to $\phi(x)$, and therefore to $\phi^n(x)$ for any $n \geq 0$.
3. Show that two elements $x, y \in G$ are ϕ–conjugate if and only if xz and yz are conjugate (in the usual sense) in $M(\phi)$. Hint: use the fact that an arbitrary element of $M(\phi)$ can be written in the form γz^n for some $\gamma \in G$.

Problem 2. Let α, β be two orientation–preserving symmetries (not necessarily isometries!) of $\mathbb{E}^2$ which fix the origin.

1. Show that α, β generate an abelian group $G = \langle \alpha, \beta \rangle$.
2. Under what circumstances is there a point $p \in \mathbb{E}^2$ whose orbit under G is dense?

Problem 3. Let $\alpha(t)$ be a smooth curve on the sphere, parameterized by arclength and suppose $x_0 \neq \alpha(t_0)$ is another point in the sphere. Prove that the derivative of the spherical distance $d(x_0, \alpha(t))$ with respect to t at t_0 is $\cos(\theta)$ where θ is the angle between the geodesic from x_0 to $\alpha(t_0)$ and $\alpha'(t_0)$, provided $\alpha(t_0)$ and x_0 are not antipodal.

Problem 4. Prove the "law of sines for right–angled hyperbolic hexagons": if a, b, c are the lengths of alternate sides of a right–angled hyperbolic convex hexagon, and a', b', c' are the lengths of the opposite sides, then

$$\frac{\sinh a}{\sinh a'} = \frac{\sinh b}{\sinh b'} = \frac{\sinh c}{\sinh c'}$$

Problem 5. Let Σ be a surface of genus 2.

1. Find a degree 2 covering of Σ by a surface Σ' of genus 3.
2. Consequently, find an injective homomorphism of $\pi_1(\Sigma')$ into $\pi_1(\Sigma)$. Describe this homomorphism in terms of the image of generators for a "standard" presentation of these groups, and show that the homomorphism is well–defined — that

is, if w is a relation in the generators of $\pi_1(\Sigma')$, the image of w in $\pi_1(\Sigma)$ can be trivialized by the relations in the presentation of $\pi_1(\Sigma)$

Answer only one of the following two problems

Problem 6. Suppose Σ is a genus 2 hyperbolic surface.

(1) Show that there is an orientation–preserving isometry $i : \Sigma \to \Sigma$ such that $i^2 = \text{id}$ (one says i is an *involution*) with 6 fixed points.
(2) Identify the quotient orbifold Σ/i.
(3) Show that there are hyperbolic surfaces of genus 3 which do not admit any orientation–preserving isometries.

Problem 7. Let Σ be a closed oriented hyperbolic surface, and γ a simple closed geodesic.

(1) Sketch the effect of the Dehn twist τ_γ on the universal cover $\widetilde{\Sigma} = \mathbb{H}^2$.
(2) Show that infinitely many points in S^1_∞ are kept pointwise fixed, but that other points on S^1_∞ are moved. Deduce that τ_γ is not equivalent to an isometry in $\text{MC}(\Sigma)$, and therefore that a Dehn twist acts on $\mathcal{MH}(\Sigma)$ without fixed points.

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