Practice Problems

Here are some practice problems for the midterm:

1. Give an example of a sequence of functions \( \{f_n\}_{n \in \mathbb{N}} \) where \( f_n : [0, 1] \to \mathbb{R} \) which does not converge uniformly. Prove your claims either directly or through consequences of the theorems we have proved in class.

2. Suppose that \( \{f_n\}_{n \in \mathbb{N}} \) from \([0, 1] \to [0, 1]\) is a sequence of continuous functions such that

\[
\max_{0 \leq t \leq 1} |f_{n+1}(t) - f_n(t)| \leq \frac{1}{2} \max_{0 \leq t \leq 1} |f_n(t) - f_{n-1}(t)|.
\]

Prove that

\[
\max_{0 \leq t \leq 1} |f_{n+1}(t) - f_n(t)| \leq 2^{-(n-1)}
\]

and that \( \{f_n\}_{n \in \mathbb{N}} \) converges uniformly to a continuous function \( f : [0, 1] \to [0, 1] \).

3. (a) State the Weierstrass M-test.
   (b) Consider

\[
f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.
\]

For what \( x \in \mathbb{R} \) is \( f(x) \) a continuous function.

4. Consider the sequence of functions

\[
f_n(x) = \frac{nx}{1 + nx}
\]

for \( x \in [0, \infty) \). Does the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converge uniformly on \([0, 1]\)? On \([1, \infty)\)?

5. (a) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be continuous. Let \( U \subset \mathbb{R}^m \) be a closed set. Prove that \( f^{-1}(U) \) is closed in \( \mathbb{R}^n \).
   (b) Give an example of a continuous function \( f : \mathbb{R}^n \to \mathbb{R}^m \) and a closed set \( V \subset \mathbb{R}^n \) such that \( f(V) \) is not closed.

6. Let \( K \subset \mathbb{R}^n \) be compact and \( f : K \to \mathbb{R} \) a continuous function. Prove that \( f \) is uniformly continuous.

7. Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation. Prove that the following are equivalent:
   (a) \( T \) is continuous.
   (b) \( T \) is continuous at 0.
   (c) \( T \) is bounded, i.e. there exists a \( C > 0 \) such that for all \( x \in \mathbb{R}^n \), \(|T(x)| \leq C|x|\).

8. Suppose that \( f, g : \mathbb{R}^n \to \mathbb{R}^m \) are continuous functions which agree on a dense set. Prove that \( f = g \).
9. Let $A, B \subseteq \mathbb{R}^n$, $A \cap B = \emptyset$. Define $f : \mathbb{R}^n \to \mathbb{R}^m$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$ 

Prove that $f$ is continuous, and $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.

10. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be independent of $x^1, \ldots, x^j$.

   (a) Prove that $f(x) = g(x^{j+1}, \ldots, x^n)$ for some $g : \mathbb{R}^{n-j} \to \mathbb{R}^m$.

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$. Prove that $f$ is continuous at $a$.

   (c) Let $a \in \mathbb{R}^n$ and suppose that $f$ is differentiable at $a$. Compute $Df(a)$ in terms of $g$.

11. Consider $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} 
\frac{x^3 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0).
\end{cases} \quad (1)$$

Prove that $f_x(0, 0)$ and $f_y(0, 0)$ both exists but that $f$ is not differentiable at $(0, 0)$.

12. Let $f(x, y, z) = (\sin(xy \sin z), z^3)$. Compute $Df(a)$. 