Math 163 (51) - Midterm Test
Spring Quarter 2017
Friday, May 12, 2017

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets. Feel free to use the opposite side.
- This exam has 6 pages, and 5 problems. Please make sure that all pages are included.
- Each problem is worth 10 points.
- You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
- Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
Question 1.  (a) (2 points) Define uniform convergence of a sequence \( \{f_k\}_{n \in \mathbb{N}} \)， with \( A \subseteq \mathbb{R}^2 \), \( f_n : A \to \mathbb{R} \), to a function \( f : A \to \mathbb{R} \).

(b) (8 points) Let \( K \subseteq \mathbb{R}^2 \) be a compact set and let \( f_n : K \to \mathbb{R} \) be a decreasing sequence of continuous functions, i.e. \( f_n(x) \geq f_{n+1}(x) \) for all \( x \in K \) and \( n \in \mathbb{N} \). Suppose that the sequence \( \{f_n\} \) converges point-wise to zero. Prove that \( \{f_n\} \) converges uniformly to zero.
Question 2. (a) (7 points) Consider the sequence \( \{a_n\}_{n \in \mathbb{N}} \) with \( a_n = (-1)^n n \).

Define

\[
f(x) = \sum_{n=1}^{\infty} a_n x^n.
\]

Prove that \( f \) is continuous on \((-1, 1)\).

(b) (3 points) Compute \( \lim_{x \to 1^-} f(x) \). Justify each step.
Question 3.  (a) (7 points) Consider the graph of a bounded function \( f : \mathbb{R} \to \mathbb{R} \), that is, the set
\[
G_f = \{ (x, f(x)) : x \in \mathbb{R} \} \subseteq \mathbb{R}^2.
\]
Prove that if \( G_f \) is closed then \( f \) is continuous.

(b) (3 points) (Corrected statement) Let \( A \subseteq \mathbb{R} \). Give an example of a continuous function \( f : A \to \mathbb{R} \) whose graph is not closed in \( \mathbb{R}^2 \).
Question 4. Consider

\[ f(x, y) = \begin{cases} 
  \frac{x^3 - xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\
  0 & (x, y) = (0, 0). 
\end{cases} \] (1)

Prove that \( f_x(0, 0) \) and \( f_y(0, 0) \) exist but that \( f \) is not differentiable at \((0, 0)\).
Question 5. (a) (5 points) Recall that a directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction $v$ is defined by

$$D_v f(a) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.$$

Prove the following version of the Mean Value Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. For any $a, v \in \mathbb{R}^n$, there exists $t_0 \in (0, 1)$ such that

$$f(a + v) - f(a) = D_v f(a + t_0 v).$$

(b) (5 points) Let $A \subseteq \mathbb{R}^2$ have the property that for any $x, y \in A$, there exists a line which connects $x$ and $y$ in $A$. Let $f : A \to \mathbb{R}$ be continuously differentiable and suppose that for every $x \in A$, $D_1 f(x) = D_2 f(x) = 0$. Prove that $f$ is constant on $A$. 