Math 161 (31) - Midterm Test 2
Autumn Quarter 2017
Friday, November 10, 2016

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets. Feel free to use the opposite side.
- This exam has 6 pages, and 5 problems. Please make sure that all pages are included.
- Each problem is worth 10 points.
- You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
- Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
Question 1. Suppose $A$ is a non-empty, bounded subset of $\mathbb{R}$. For a fixed $c > 0$, let

$$cA = \{ca \mid a \in A\}.$$

Prove that $\sup cA = c \sup A$.

Solution: Since $A$ is non-empty and bounded, $c > 0$, $cA$ is non-empty and bounded, so $\sup cA$ exists. Denote $\sup A = \alpha$. Since $\alpha$ is an upper bound for $A$ this implies that

$$a \leq \alpha, \quad \text{for all } a \in A,$$
and hence since $c > 0$,

$$ca \leq c\alpha, \quad \text{for all } a \in A,$$
so $c\alpha$ is an upper bound for $cA$. Now, fix $\delta > 0$, then there exists $a_\delta$ such that

$$a_\delta > \alpha - \frac{\delta}{c}.$$

Hence, since $c > 0$, we have

$$ca_\delta > c\alpha - c \cdot \frac{\delta}{c} = c\alpha - \delta,$$
and we obtain the desired conclusion since $ca_\delta \in cA$. 
Question 2. Prove that the set

\[ A = \{ x \in \mathbb{Q} \mid x^2 < 3 \} \]

is non-empty and bounded above, but has no least upper bound in \( \mathbb{Q} \). (You may use that \( \sqrt{3} \notin \mathbb{Q} \), and that \( \mathbb{Q} \) is dense).

Solution: Proof done in class for

\[ A = \{ x \in \mathbb{Q} \mid x^2 < 2 \}. \]
Question 3.  1. (3 points) Let $A \subseteq \mathbb{R}$. Define uniform continuity of a function $f : A \to \mathbb{R}$.

**Solution:** A function $f$ is uniformly continuous on $A$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

2. (7 points) Prove that the function $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

**Solution:** Fix $\varepsilon > 0$. Since $\sqrt{x}$ is continuous on $[0, 2]$ it is uniformly continuous there, so let $\delta_1 > 0$ be such that for all $x, y \in [0, 2]$, $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \varepsilon$. We also showed in class that $\sqrt{x}$ is Lipschitz on $[1, \infty)$, so it is uniformly continuous there and let $\delta_2 > 0$ be such that for all $x, y \in [1, \infty)$, $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \varepsilon$. Then take $\delta = \min(\delta_1, \delta_2, 1)$. 
Question 4. Suppose that \( f \) is continuous on \( \mathbb{R} \) and
\[
\lim_{x \to \infty} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = -\infty
\]
Prove that \( f(x) = 0 \) for some \( x \in \mathbb{R} \).

Solution: Fix \( M > 0 \). By definition of limits, we can find \( N_1 < 0 < N_2 \) such that for all \( x < N_1 \),
\[
f(x) < -M
\]
and for all \( x > N_2 \),
\[
f(x) > M.
\]
Hence
\[
f(N_1 - 1) < -M < 0 < M < f(N_2 + 1),
\]
and we can apply the intermediate value theorem to \( f \) on the interval \([N_1 - 1, N_2 + 1]\).
Question 5.  

1. (4 points) Let \( f \) be a continuous function on \([a, b] \subseteq \mathbb{R} \). Define \( f^*(x) = \sup\{f(y) \mid a \leq y \leq x\} \).

Prove that \( f^*(x) \) is a function on \([a, b] \).

Solution: Since \( f \) is continuous on \([a, b] \) it is continuous on \([a, x] \) for any \( x \in [a, b] \), and hence it is bounded, so for every \( x \in [a, b] \),

\[ \{f(y) \mid a \leq y \leq x\} \]

is non-empty and bounded above, so a supremum exists, and by a result from class it is unique. Thus for all \( x \in [a, b] \), there exists a unique value assigned to \( f^*(x) \), so it is a function.

2. (6 points) Prove that \( f^*(x) \) is increasing and continuous on \([a, b] \). (Hint: how much can the supremum of a continuous function change for \( x \) and \( z \) which are sufficiently close).

Solution: For \( x \leq z \), we have

\[ \{f(y) \mid a \leq y \leq x\} \subseteq \{f(y) \mid a \leq y \leq z\} \]

hence

\[ \sup\{f(y) \mid a \leq y \leq x\} \leq \sup\{f(y) \mid a \leq y \leq z\}, \]

so \( f^*(x) \leq f^*(y) \). To prove that this function is continuous, fix \( z \in [a, b] \) and \( \varepsilon > 0 \). Let \( \delta > 0 \) be such that for all \( |x - z| < \delta \) we have \( |f(x) - f(z)| < \varepsilon / 2 \). Then this implies that for all \( x - z < \delta \),

\[ |f(x)| < |f(z)| + \varepsilon / 2 \]

and hence,

\[ f^*(x) \leq f^*(z) + \varepsilon / 2. \]

and similarly, for \( z - x < \delta \),

\[ f^*(z) \leq f^*(x) + \varepsilon / 2. \]

Hence

\[ |f^*(x) - f^*(z)| \leq \varepsilon / 2 < \varepsilon. \]