Math 161 (31) - Midterm Test 2
Autumn Quarter 2017
Friday, November 10, 2016

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets. Feel free to use the opposite side.
- This exam has 6 pages, and 5 problems. Please make sure that all pages are included.
- Each problem is worth 10 points.
- You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
- Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
**Question 1.** Suppose $A$ is a non-empty, bounded subset of $\mathbb{R}$. For a fixed $c > 0$, let

$$cA = \{ca \mid a \in A\}.$$ 

Prove that $\sup cA = c \sup A$. 

Question 2. Prove that the set

\[ A = \{ x \in \mathbb{Q} \mid x^2 < 3 \} \]

is non-empty and bounded above, but has no least upper bound in \( \mathbb{Q} \). (You may use that \( \sqrt{3} \notin \mathbb{Q} \), and that \( \mathbb{Q} \) is dense).
Question 3.  

1. (3 points) Let $A \subseteq \mathbb{R}$. Define uniform continuity of a function $f : A \to \mathbb{R}$.

2. (7 points) Prove that the function $\sqrt{x}$ is uniformly continuous on $[0, \infty)$. 

Question 4. Suppose that \( f \) is continuous on \( \mathbb{R} \) and

\[
\lim_{x \to \infty} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = -\infty
\]

Prove that \( f(x) = 0 \) for some \( x \in \mathbb{R} \).
Question 5.  
1. (4 points) Let $f$ be a continuous function on $[a, b] \subseteq \mathbb{R}$. Define

$$f^*(x) = \sup \{ f(y) \mid a \leq y \leq x \}.$$

Prove that $f^*(x)$ is a function on $[a, b]$.

2. (6 points) Prove that $f^*(x)$ is increasing and continuous on $[a, b]$. (Hint: how much can the supremum of a continuous function change for $x$ and $z$ which are sufficiently close).