Math 161 (51) - Midterm Test 1  
Autumn Quarter 2016  
Friday, October 21, 2016

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets. Feel free to use the opposite side.
- This exam has 7 pages, and 5 problems. Please make sure that all pages are included.
- Each problem is worth 10 points. The first problem has 5 true/false questions worth 2 points each.
- You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
- Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
Question 1. (True/False.)

1. The function \( f : \mathbb{R} \to \mathbb{R} \) given by
\[
  f(x) = \begin{cases} 
    0 & x \text{ irrational} \\
    \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \text{ (for } \frac{p}{q} \text{ in lowest terms, } q > 0) 
  \end{cases}
\]
is periodic.

2. Suppose that \( g : \mathbb{R} \to \mathbb{R} \) satisfies \( g(x) > 0 \) for all \( x \). If \( \lim_{x \to a} g(x) \) does not exist then \( \lim_{x \to a} \left( \frac{1}{g(x)} \right) \) does not exist.

3. The fact that \( \sqrt{2} \) is irrational can be deduced from P1 - P12.

4. The well-ordering property of the natural numbers implies the principle of complete induction.

5. The subset of points \((x, y)\) in \( \mathbb{R}^2 \) satisfying the condition \( x = |y| \) is the graph of a function.

Solution.

1. T
2. F
3. F
4. T
5. F
Question 2. Prove using only P1 - P12, indicating where you are using each property, that if \( a > 2 \) then \( a^2 > 4 \).

Solution. By definition \( a > 2 \) implies \( a - 2 > 0 \). Since \( a - 2 > 0 \) and \( 2 > 0 \), \( a - 2 + 2 = a > 0 \) by the property of additive inverses and by closure of positivity under addition. Hence \( a > 0 \), and thus \( a + 2 > 0 \) by closure of positivity under addition. Finally \( a^2 - 4 = (a - 2)(a + 2) > 0 \) by distributivity, and closure of positivity under multiplication.
Question 3.  a) (3 points) State the principle of mathematical induction. Solution.

Let $P(n)$ denote that a property $P$ is true for a natural number $n$. Then the principle of mathematical induction states that if

(a) $P(0)$ is true,

(b) and $P(n)$ implies $P(n+1)$,

then $P(n)$ is true for all $n \in \mathbb{N}$.

b) (7 points) Prove that $1 + \ldots + n = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

Solution. Let $P(n)$ denote the property that the above formula holds for $n \in \mathbb{N}$. We first check that $P(0)$ is true, and indeed $0 = 0$. Now suppose that $P(n)$ is true and consider

$$1 + \cdots + n + n + 1.$$  

(1)

By the inductive hypothesis

$$1 + \cdots + n + n + 1 = \frac{n(n + 1)}{2} + n + 1 = \frac{n(n + 1)}{2} + \frac{2(n + 1)}{2} = \frac{(n + 2)(n + 1)}{2}$$  

(2)

and hence $P(n)$ implies $P(n + 1)$ and by induction, the formula holds for all $n \in \mathbb{N}$. 
Question 4. Consider functions $f : X \to Y$ and $g : Y \to Z$. Prove that

$$h = \{(x, z) : \text{there exists } y \text{ such that } (x, y) \in f \text{ and } (y, z) \in g\} \subseteq X \times Z$$

is a function. What is its domain and codomain?

Solution. We claim that this set is a function with domain $X$ and codomain $Z$. We need to show that for every $x \in X$, there is a unique $z \in Z$ so that $h(x) = z$, or, in other words, every $x \in X$ appears precisely once as the first coordinate of an ordered pair $(x, z) \in h$. Fix any $x \in X$. Then, since $x$ is in the domain of $f$, there is a unique $y \in Y$ such that $f(x) = y$. Now, since the domain of $g$ is $Y$, for this $y \in Y$, there is a unique $z \in Z$ such that $g(y) = z$. So for every $x \in X$, there exists precisely one $y \in Y$ such that $(x, y) \in f$ and $(y, z) \in g$, and since $g$ is a function, this choice of $z$ is unique.
Question 5.  

a) (2 points) State the \( \varepsilon - \delta \) definition of a limit.

**Solution.** We say \( \lim_{x \to a} f(x) = l \) if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
0 < |x - a| < \delta, \quad |f(x) - l| < \varepsilon.
\]

b) (5 points) Let \( f, g : \mathbb{R} \to \mathbb{R} \). Prove that if \( \lim_{x \to a} f(x) = 0 \) and \( g \) is bounded for all \( x \) in a neighborhood of \( a \) then \( \lim_{x \to a} (fg)(x) = 0 \).

**Solution.** Fix \( \varepsilon > 0 \). We need to show there exists \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \), then \( |(fg)(x)| < \varepsilon \). We have

\[
|(fg)(x)| = |f(x)||g(x)| \leq M|f(x)|
\]

by the hypotheses on the boundedness of \( g \). Thus, fix \( \delta > 0 \) so that if \( 0 < |x - a| < \delta \), \( |f(x)| < \varepsilon/M \), then for \( 0 < |x - a| < \delta \), we have

\[
|(fg)(x)| \leq M|f(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon,
\]

as required.
c) (3 points) Prove, using the $\varepsilon - \delta$ definition of a limit, that
\[
\lim_{x \to 0} (x^2 + 1) \sin x = 0.
\]
(Hint: You may use that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$).

**Solution.** Fix $\varepsilon > 0$. Fix $\delta_1 = 1$, then if $0 < |x| < 1$,
\[
x^2 + 1 \leq 2,
\]
hence $x^2 + 1$ is bounded in a neighbourhood of 0. Now, since $|\sin(x)| \leq |x|$, fix $\delta_2 = \varepsilon > 0$, hence for $0 < |x| < \delta_2$
\[
|\sin(x)| \leq |x| < \varepsilon,
\]
so $\lim_{x \to 0} \sin x = 0$. Thus, from part (b) of this problem, with $f(x) = \sin x$ and $g(x) = x^2 + 1$ we obtain that
\[
\lim_{x \to 0} (x^2 + 1) \sin x = 0.
\]