Math 162 (31) - Midterm Test 1
Winter Quarter 2018
Wednesday, February 7, 2018

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets. Feel free to use the opposite side.
- This exam has 6 pages, and 5 problems. Please make sure that all pages are included.
- Each problem is worth 10 points.
- You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
- Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
Question 1. Let $f$ and $g$ be integrable on $[a, b]$ and suppose that $f(x) \leq g(x)$, $x \in [a, b]$ prove that

$$\int_a^b f \leq \int_a^b g.$$  

Using this, show that for any integrable function $f$, we have

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$
Question 2. Let $f : [0, b] \to \mathbb{R}$ be a continuous monotone function. Prove that $f$ is integrable on $[0, b]$. Hint: consider the partitions $P_n = \{ \frac{ib}{n}, \ i = 0, 1, \ldots, n \}$. 
Question 3. Show that

\[ \int_1^n \log x \, dx = n \log n - n + 1. \]

By considering a certain upper sum, use this to show that

\[ \log(n!) \geq n \log n - n + 1 \]

and deduce a lower bound for \( n! \) from this inequality. Recall \( n! = n \cdot (n - 1) \cdots 2 \cdot 1 \).
Question 4. Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a non-negative smooth function such that \( \phi^{(k)}(x) = 0 \) for \( |x| \geq 1 \) and \( k \geq 0 \) and suppose that \( \int_{-1}^{1} \phi(x) = 1 \). Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is an \( n \)-times continuously differentiable function. Prove that

\[
\lim_{h \to 0} \int_{-1}^{1} \frac{1}{h^{k+1}} \phi^{(k)} \left( \frac{x}{h} \right) f(x) \, dx = (-1)^k f^{(k)}(0),
\]

for \( k = 0, 1, \ldots, n \) without relying on the related homework problem. Hint: integrate by parts.
Question 5.  (a) (5 points) Let $g$ be continuous on $\mathbb{R}$. Prove that

$$y(x) = \int_a^x g(t) \, dt + C \quad \text{if and only if} \quad y'(x) = g(x).$$

(b) (5 points) Suppose again that $g$ is continuous, and that $f$ is non-zero, differentiable, and

$$f(x) = \int_a^x \frac{g(t)}{f^2(t)} \, dt + 1.$$

Use part (a) to determine a formula for $f(x)$. Justify your work!