Instructions:

• Read each problem carefully.
• Write legibly.
• Show all your work on these sheets. Feel free to use the opposite side.
• This exam has 6 pages, and 5 problems. Please make sure that all pages are included.
• Each problem is worth 10 points.
• You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
• Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
Question 1. Prove using only P1 - P12, indicating where you are using each property, that if $0 < a < 1$ then $a^2 < a$.

Solution: By definition, $a > 0$ and $(1 - a) > 0$. By closure under multiplication, $a(1 - a) > 0$ and hence, by distributivity, $a - a^2 > 0$, which yields the result by definition.
Question 2.  

a) (3 points) State the principle of mathematical induction.

Solution: Let \( P(x) \) be the statement that a property \( P \) holds for a natural number \( x \). If \( P(0) \) is true, and whenever \( P(k) \) is true, \( P(k+1) \) is true, then \( P(x) \) holds for all \( x \in \mathbb{N} \).

b) (7 points) Prove by induction on \( n \) that

\[
1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}.
\]

Solution: The base case (with \( n = 0 \)) holds, so suppose this statement holds for some natural number \( n \geq 1 \), we will show this implies the statement for \( n + 1 \). Indeed, by the inductive hypothesis

\[
1 + r + r^2 + \ldots + r^n + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1} = \frac{1 - r^{n+1} + r^{n+2} - r^{n+2}}{1 - r} = \frac{1 - r^{(n+1)+1}}{1 - r}.
\]
Question 3. Prove that if sets $A$ and $B$ are countable then $A \times B$ is countable.

Solution: Since $A$ and $B$ are countable, we can enumerate the elements of these sets by $(a_1, a_2, \ldots)$, and $(b_1, b_2, \ldots)$. Consider now an array where the element $(a_i, b_j)$ is located in the $i$-th row and the $j$-th column, now use the same procedure as for $\mathbb{Q}$. 
Question 4.  

(a) (3 points) State the definition of a function $f : A \to B$.

**Solution:** A function $f$ with domain $A$ and codomain $B$ is a subset of the cartesian product $A \times B$ such that every $a \in A$ appears exactly once as the first coordinate in an ordered pair $(a,b)$ in $f$.

(b) (7 points) Suppose that $(f \circ g)(x) = x$ for all $x \in \mathbb{R}$. Prove that $g$ is one-one and $f$ is onto.

**Solution:** First we note that the range of $f$ is necessarily the range of the function $x$, which is all of $\mathbb{R}$, hence $f$ is onto. To show that $g$ is one-one, suppose that $g(x_1) = g(x_2)$, we will show that $x_1 = x_2$. Indeed, for such $x_1, x_2$, we have

$$(f \circ g)(x_1) = f(g(x_1)), \quad (f \circ g)(x_1) = f(g(x_2)).$$

Since $g(x_1) = g(x_2)$ and $f$ is a function, we must have $f(g(x_1)) = f(g(x_2))$. Using the hypothesis on $f \circ g$, we then conclude that

$$x_1 = (f \circ g)(x_1) = (f \circ g)(x_2) = x_2,$$

as required.
Question 5.  

a) (2 points) State the $\varepsilon - \delta$ definition of a limit.

**Solution:** A function $f$ approaches a limit $l$ at $a$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - l| < \varepsilon$.


b) (6 points) Define

$$f(x) = \begin{cases} x, & \text{x irrational} \\ -x, & \text{x rational} \end{cases}$$

Prove, using the $\varepsilon - \delta$ definition of a limit, that

$$\lim_{x \to 0} f(x) = 0.$$

but that $\lim_{x \to a} f(x)$ does not exist for $a \neq 0$. (Hint: For $a \neq 0$, there exists $\varepsilon > 0$ such that $|a| > \varepsilon > 0$.)

**Solution:** Fix $\varepsilon > 0$. First we note that at 0,

$$|f(x) - 0| = |x|,$$

so we conclude that $\lim_{x \to 0} f(x) = 0$ by choosing $\delta = \varepsilon$. Now let $a \neq 0$ and suppose (by symmetry) $a > 0$. Fix $\varepsilon = a/4$. Then for all $\delta > 0$, by density, there exist both rational and irrational numbers in the $\delta$-neighbourhood of $a$.

We consider $x_\delta$ rational, with $0 < |x_\delta - a| < \min(\delta, \frac{a}{2})$, then $x_\delta > \frac{a}{2}$ and

$$|\ell - f(x_\delta)| = \ell + x_\delta \geq x_\delta \geq \frac{a}{2} > \varepsilon.$$ 

Similarly, if $\ell < 0$ we fix the same $\varepsilon > 0$ and consider $x_\delta$ irrational with $0 < |x_\delta - a| < \min(\delta, \frac{|a|}{2})$ and we obtain the same result.