Write clearly, and use a different page for each problem. You are encouraged to work together but problems should be written up individually. No late homework will be accepted. All problem numbers are from Linear Algebra Done Wrong. Numbering will take the form “m.n.r” where m is the chapter, n is the subsection, and r is the problem number. Do the following problems:

1. a) Use an augmented matrix and Gauss-Jordan elimination to solve the following system of equations:
   \[
   \begin{align*}
   3x + 2y + z + 6w &= -1 \\
   -2x + 3y + (-2z) + 2w &= 5.
   \end{align*}
   \]
   Express the solution set as a particular solution plus the set of solutions to the corresponding homogeneous system.

b) Prove that if a system of equations with real coefficients which has more variables than equations has a single solution, then it has infinitely many solutions.

2. Use Gauss-Jordan elimination to find the inverse of the following invertible matrices \( A \in M_n(\mathbb{R}) \) as follows. Apply the method of row reduction to the \( n \times 2n \) augmented matrix \([ A \mid I]\) to get \([ I \mid A^{-1}]\), where \( I \in M_n(\mathbb{R}) \) is the identity matrix.

   \[
   (a) \quad A = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 1 & -2 \\ 0 & 3 & 4 \end{bmatrix} \quad \text{and} \quad (b) \quad A = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}
   \]

3. Use Gauss Jordan elimination to find a basis for the subspace of \( \mathbb{R}^5 \) spanned by the following vectors:
   \[
   u_1 = (4, 3, 2, 1, 0) \\
   u_2 = (1, 0, 3, 1, 2) \\
   u_3 = (2, 1, 0, 1, 2) \\
   u_4 = (0, 1, 1, 1, 0) \\
   u_5 = (3, 0, 2, 1, 4).
   \]

4. (Polynomial interpolation) This is an application of the rank-nullity theorem to the problem of finding a polynomial which passes through given points. Let \( x_0, y_0, \ldots, x_n, y_n \in \mathbb{R} \) be distinct.

   a) Consider \( \mathbb{P}_k \) the space of polynomials of degree \( \leq k \). Write down a system of equations (in matrix form) whose solutions are polynomials \( p \) satisfying
   \[
   p(x_i) = y_i, \quad \text{for all } i = 0, \ldots, n. \tag{1}
   \]
   For which \( k \) can you guarantee the existence of a solution to this system? For which \( k \), is the solution unique?
b) By considering the map

\[ T: \mathbb{P}_n \to \mathbb{R}^{n+1}, \quad p \mapsto (p(x_0), \ldots, p(x_n)), \]

use the rank-nullity theorem to give an alternative proof that there exists a polynomial of degree \( \leq n \) satisfying (1).