Homework 4 – Due Monday February 13

Write clearly, and use a different page for each problem. You are encouraged to work together but problems should be written up individually. No late homework will be accepted. All problem numbers are from the Fourth Edition of Spivak’s Calculus. I’m writing out the problems since I didn’t have the proper numbering – they’re mostly from Spivak and I will add the numbering this Monday. Due by class in my box in the basement of Eckhart.

• Practice problems on integrals (not to be handed in): 19.1-19.11
• Suppose that $f''$ is continuous and that

$$\int_{0}^{\pi} [f(x) + f''(x)] \sin x \, dx = 2.$$ 

Given that $f(\pi) = 1$, compute $f(0)$.
• Express $\int \log(\log x) \, dx$ in terms of $\int (\log x)^{-1} \, dx$.
• Do all parts of this problem: Let $\phi$ be a nonnegative integrable function with $\phi(x) = 0$ for $|x| \geq 1$ and such that $\int_{-1}^{1} \phi = 1$. For $h > 0$ let

$$\phi_h(x) = \frac{1}{h} \phi(x/h).$$

(a) Show that $\phi_h(x) = 0$ for $|x| \geq h$ and that $\int_{-h}^{h} \phi_h = 1$.
(b) Let $f$ be integrable on $[-1,1]$ and continuous at 0. Show that

$$\lim_{h \to 0^+} \int_{-1}^{1} \phi_h f = \lim_{h \to 0^+} \int_{-h}^{h} \phi_h f = f(0).$$

(c) Show that

$$\lim_{h \to 0^+} \int_{-1}^{1} \frac{h}{h^2 + x^2} \, dx = \pi.$$ 

(d) Let $f$ be integrable on $[-1,1]$ and continuous at 0. Show that

$$\lim_{h \to 0^+} \int_{-1}^{1} \frac{h}{h^2 + x^2} f(x) \, dx = \pi f(0).$$

• If $f'$ is continuous on $[a,b]$, use integration by parts to prove the Riemann-Lebesgue Lemma for $f$:

$$\lim_{\lambda \to \infty} \int_{a}^{b} f(t) \sin(\lambda t) \, dt = 0.$$
Suppose that \( f \) is integrable on \([a, b]\) and that \( \phi \) is either nondecreasing or nonincreasing on \([a, b]\). Then there is a number \( \xi \) in \([a, b]\) such that

\[
\int_a^b f(x)\phi(x)dx = \phi(a)\int_a^b f(x)dx + \phi(b)\int_\xi^b f(x)dx.
\]

(a) Prove the result is true for nonincreasing \( \phi \), then it is also true for nondecreasing \( \phi \).

(b) Prove that if the result is true for nonincreasing \( \phi \) satisfying \( \phi(b) = 0 \) then it is true for all nonincreasing \( \phi \). Thus we only have to prove

\[
\int_a^b f(x)\phi(x)dx = \phi(a)\int_a^b f(x)dx.
\]

(c) Prove this using integration by parts.

(d) Show by examples that the hypothesis that \( \phi \) is either nondecreasing or nonincreasing is needed.

This is a question about the gamma function

\[ \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \]

(a) Prove that the improper integral is defined if \( x > 0 \).

(b) Prove that \( \Gamma(x + 1) = x\Gamma(x) \).

(c) Show that \( \Gamma(1) = 1 \) and hence \( \Gamma(n) = (n - 1)! \) for all natural numbers \( n \).

(a) Suppose that \( \frac{f(x)}{x} \) is integrable on every interval \([a, b]\) for \( 0 < a < b \) and \( \lim_{x \to 0} f(x) = A \) and \( \lim x \to 0 f(x) = B \). Prove for all \( \alpha, \beta > 0 \) that

\[
\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}dx = (A - B)\log \frac{\beta}{\alpha}.
\]

(b) Now suppose that \( f(x) \) converges for all \( a > 0 \) and that \( \lim_{x \to 0} f(x) = A \). Prove that

\[
\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}dx = A\log \frac{\beta}{\alpha}.
\]

1. (c) Compute the following integrals:

(i) \[ \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}dx \]

(ii) \[ \int_0^\infty \frac{\cos(\alpha x) - \cos(\beta x)}{x}dx \]

(Bonus) Prove that

\[ \int_{-\infty}^\infty e^{-x^2} = \sqrt{\pi}. \]

Hint: Show that

\[ \left( \int_{-\infty}^\infty e^{-x^2} \right)^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2 - y^2}dxdy \]

and switch to polar coordinates.