Write clearly, and use a different page for each problem. You are encouraged to work together but problems should be written up individually. No late homework will be accepted. The numbered problems are from Spivak’s Calculus on Manifolds.

- a) For \( x, y \in \mathbb{R}^n \), define

\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.
\]

Prove that \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\) is an inner product space. You should also check that \( \mathbb{R}^n \) together with + and \( \cdot \) as we defined them in class is a real vector space, but you do not need to hand that in.

- b) Let \( a_1, \ldots, a_n \in \mathbb{R} \). For \( x, y \in \mathbb{R}^n \), define

\[
\langle x, y \rangle_a = \sum_{i=1}^{n} a_i x_i y_i.
\]

Prove that \( \langle \cdot, \cdot \rangle_a \) is an inner product on \( \mathbb{R}^n \) if and only if \( a_i > 0 \) for all \( 1 \leq i \leq n \).

- Let \( \langle \cdot, \cdot \rangle \) be an inner product on a vector space \( V \). Prove that

\[
\|v\| = \sqrt{\langle v, v \rangle}, \quad v \in V
\]

defines a norm on \( V \). We say that such a norm is induced by an inner product.

- Prove that a norm is induced by an inner product if and only if it satisfies the parallelogram law:

\[
(\|x + y\|^2 + \|x - y\|^2) = 2(\|x\|^2 + \|y\|^2).
\]

- For \( 1 \leq p < \infty \) define the function \(| \cdot |_p : \mathbb{R}^n \to \mathbb{R} \) by

\[
|x|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.
\]

and define \(| \cdot |_\infty : \mathbb{R}^n \to \mathbb{R} \) by

\[
|x|_\infty = \max_{1 \leq i \leq n} |x_i|.
\]

(a) Use Young’s inequality, namely that for \( a, b \in \mathbb{R} \),

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}
\]

provided \( \frac{1}{p} + \frac{1}{q} = 1 \), to prove Hölder’s inequality: for any \( x, y \in \mathbb{R}^n \),

\[
|\langle x, y \rangle| \leq \|x\|_p \|y\|_q
\]

provided \( \frac{1}{p} + \frac{1}{q} = 1 \). Note that the case \( p = q = 2 \) is the Cauchy-Schwarz inequality. Hint: start with the case \( |x|_p = |y|_q = 1 \).
(b) Prove that $|·|_p$ is a norm for any $1 \leq p \leq \infty$. Hint: Use Hölder’s inequality to prove the triangle inequality.

(c) Draw the unit sphere for $p = 1, 2, 4, \infty$, that is draw

$$\{x \in \mathbb{R}^n : |x|_p = 1\}.$$

**Remark 1.** The case $p = 2$ is the norm we saw in class and is called the Euclidean norm. The norms $|·|_p$ are called $\ell^p$ norms.

• Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Let $e_1, \ldots, e_n$ be the standard basis on $\mathbb{R}^n$. Suppose that $T(e_i) = S(e_i)$ for all $1 \leq i \leq n$. Prove that $T(x) = S(x)$ for all $x \in \mathbb{R}^n$. Hence a linear transformation $T$ is uniquely determined by where it maps the standard basis.

**Remark 2.** Consequently, any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ may be represented as an $m \times n$ matrix

$$T(x) = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

(1)

where the coefficients $a_{ij}$ are uniquely determined by the relation $T(e_i) = \sum_{j=1}^{m} a_{ji} e_j$.

• 1.1
• 1.7
• 1.10
• 1.13