Homework 2 – Due April 14

Write clearly, and use a different page for each problem. You are encouraged to work together but problems should be written up individually. No late homework will be accepted. All problem numbers are from Linear Algebra Done Wrong. Numbering will take the form “m.n.r” where m is the chapter, n is the subsection, and r is the problem number. Do the following problems:

1. Let \( p \) be a prime number, and let \( \mathbb{F}_p \) denote the set of congruence classes of integers modulo \( p \). Define

\[
+ : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p, \quad [a]_p + [b]_p = [a + b]_p
\]

\[
\cdot : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p, \quad [a]_p \cdot [b]_p = [a \cdot b]_p
\]

Then \( \mathbb{F}_p \) is a field (bonus: prove this, the difficult part is proving the property about inverses for which you need that \( p \) is prime).

Now let \( V = \mathbb{F}_p^2 \) be the vector space of ordered pairs of elements of \( \mathbb{F}_p \).

(a) How many vectors are there in the vector space \( V \)?

(b) How many 0-dimensional subspaces does \( V \) have? Explain.

(c) How many 2-dimensional subspaces does \( V \) have? Explain.

(d) How many 1-dimensional subspaces does \( V \) have? Explain. Write down a complete list in the case \( p = 7 \).

This is the interesting part of the question. Note that there is an exact numerical answer that depends on \( p \). Hints: 1. Consider the subspaces \( W = \text{span}\{w\} \) where \( w \in V \) is a single vector. 2. Are any of the subspaces from Hint 1 the same?

The reason we use a finite field for this question is that it allows us to consider the complete list of such subspaces. Note that if we had asked the same question for \( V = \mathbb{R}^2 \), the number of 1-dimensional subspaces would clearly be infinite.

2. Let \( V \) be a vector space over a field \( \mathbb{F} \), and suppose \( U \) and \( W \) are subspaces of \( V \).

(a) Show that \( U \cap W \) is a subspace of \( V \).

(b) Show by example that \( U \cup W \) is not a vector space.

(c) Show that \( U + W = \{u + w \mid u \in U, w \in W\} \) is a subspace of \( V \).

3. Consider the vector space \( \mathbb{P}_3 \) of polynomials of degree less than or equal to 3:

\[
\mathbb{P}_3 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\},
\]

and consider the differentiation operator \( D : \mathbb{P}_3 \to \mathbb{P}_3 \) defined by \( D(p(x)) = p'(x) \).

(a) Find the matrix for \( D \) with respect to the basis \( \mathfrak{B}_1 = \{1, x, x^2, x^3\} \) for \( \mathbb{P}_3 \). (Use \( \mathfrak{B}_1 \) as the basis for both the domain and codomain.)
(b) Find the matrix for $D$ with respect to the basis $\mathfrak{B}_2 = \{1 - x, 1 + x, x^2 - x^3, x^2 + x^3\}$. (Use $\mathfrak{B}_2$ as the basis for both the domain and codomain.)

(c) Find the coordinates for the vector $p(x) = x^3 - 12x$ in terms of each of the bases above, and compute $D(p(x))$ by multiplying the matrix for $D$ times the coordinate vector for $p(x)$ in each of the two cases.

4. 1.5.4

5. Do the following:
   - 1.5.5
   - 1.5.7

6. 1.5.6

7. 1.6.6 – 1.6.9

8. 1.6.10

9. Let $\mathbb{F}$ be a field. Prove there exists a bijection between the vector space of linear transformations from $\mathbb{F}^n \to \mathbb{F}^m$ and $\mathcal{M}_{m,n}(\mathbb{F})$, the vector space of $m \times n$ matrices with entries in $\mathbb{F}$.

10. Let $V$ and $W$ be finite-dimensional vector spaces over $\mathbb{F}$. Let

$$\mathcal{L}(V,W) = \{L : V \to W \mid L \text{ is linear}\}.$$ 

   (a) Prove that $dim(\mathcal{L}(V,W)) = dim(V) \cdot dim(W)$.

   (b) Now suppose $dim(V) = n$ and $dim(W) = m$, and let $\{v_1, \ldots, v_n\}$ be a basis for $V$ and $\{w_1, \ldots, w_m\}$ a basis for $W$. For each $i = 1, 2, \ldots, n$ and each $j = 1, 2, \ldots, m$, define $L_{ij} : V \to W$ by the following:

$$L_{ij}(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = a_i w_j$$

Find the matrix for $L_{ij}$ with respect to the bases above.