Math 20250 - Final Exam
Spring Quarter 2021
Friday, June 4, 2021

Instructions:

• Read each problem carefully.
• Write legibly.
• Show all your work on these sheets. Feel free to use the opposite side.
• This exam has 11 pages, and 8 problems. Please make sure that all pages are included.
• Each problem is worth 10 points.
• You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
• Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.
• Throughout $P_n(\mathbb{R}) = \mathbb{P}_n$ for $n \in \mathbb{N}$ will denote the space of polynomials of degree $\leq n$ with real coefficients.

Good luck!
Question 1. (10 points) Clearly circle True or False for each statement.

True or False \( \mathbb{Z} \) is a vector space over \( \mathbb{R} \).

True or False If \( V \) is a vector space with \( \dim V = n \), then any set of \( n+1 \) vectors in \( V \) is linearly dependent.

True or False If \( A \) and \( B \) are row equivalent, then there exists an invertible matrix \( S \) such that \( B = SAS^{-1} \).

True or False Let \( A_e \) be a matrix in echelon form. The number of pivots in \( A_e \) is equal to \( \dim(\text{Ran} A_e) \).

True or False For any matrix \( A \), \( \dim(\text{Ran} A) = \dim(\text{Ker} A^T) \).

True or False If \( U, W \) are subspaces of a vector space \( V \), then \( U + W \) is also a subspace of \( V \).

True or False If \( U, W \) are subspaces of a vector space \( V \), then \( U \cup W \) is also a subspace of \( V \).

True or False A square matrix \( A \) is diagonalizable if and only if it has no zero eigenvalues.

True or False A square matrix \( A \) is invertible if and only if it has no zero eigenvalues.

True or False A matrix \( A : V \to V \) is diagonalizable if it has \( \dim(V) \) distinct eigenvalues.
Question 2.  1. Consider the system of equations

\[
\begin{align*}
    x + 2y + z + 0w &= 4 \\
    0x + y + z + 2w &= 2 \\
    0x + 2y + 2z + 4w &= 4 \\
    x + y + 0z - 2w &= 2.
\end{align*}
\]

(a) Write down the coefficient matrix for the system, call it \(A\), and find the solution set to the system (1) and a basis for \(\text{Ker}(A)\).
(b) Use your work above to find bases for $\text{Ran}(A)$ and $\text{Ran}(A^T)$, and explain how your answer agrees with the Rank Theorem.

(c) State the Rank-nullity Theorem, and explain how your answers above agree with it.

(d) Is there a value of $b \in \mathbb{R}^4$ such that $Ax = b$ has no solution? Is there a value of $b \in \mathbb{R}^4$ such that $Ax = b$ does not have a unique solution?
Question 3.

(10 pts) Let $P_2$ denote the vector space of polynomials of degree $\leq 2$ with real coefficients. Let $\mathcal{E} = \{1, x, x^2\}$ and $\mathcal{B} = \{-1, x + 1, x^2 + 1\}$. Let

$$T(\alpha + \beta x + \gamma x^2) = 3\beta + (\beta + \gamma)x + \alpha x^2.$$ 

1. Find the matrix for $T$ with respect to the basis $\mathcal{E}$.

2. Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$ coordinates, and the change of basis matrix from $\mathcal{E}$ to $\mathcal{B}$ coordinates.

3. Find the matrix for $T$ with respect to the basis $\mathcal{B}$.
Question 4. (a) Let $V$ be a finite dimensional vector space over $\mathbb{F}$ and suppose $U \subseteq V$ is a subspace. Prove that there exists a subspace $W$ such that $V = U \oplus W$.

(b) Let $V$ be a finite dimensional vector space over $\mathbb{F}$ and suppose $U$ and $W$ are subspaces of $V$. Prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$
Question 5. Consider a block matrix

\[ T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \]

(a) Provided that $A$ is invertible, establish that

\[ T = L \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} U \]

for $L$ and $U$ certain lower and upper block matrices, respectively. Use this identity to prove that

\[ \det(T) = \det(A)\det(D - CA^{-1}B). \]

(b) Prove that if $A$ is a matrix of size $n \times m$ and $B$ is a matrix of size $m \times n$ then

\[ \det(I_n + AB) = \det(I_m + BA). \]

Hint: Consider an appropriate block matrix and use the first part of this question.
(c) Generalize your proof from the previous part to show that for any \( \lambda \neq 0 \),

\[
\det(AB - \lambda I_n) = (-\lambda)^{n-m} \det(BA - \lambda I_m)
\]

and conclude that \( AB \) and \( BA \) have the same non-zero eigenvalues.
Question 6.  
a) Given a $5 \times 5$ square matrix $A$ suppose that the following operations convert $A$ to its echelon form, $A_e$ (note that numbering of rows after a swap is redefined):

- Swap $R_1$ and $R_2$
- Replace $R_3$ by $R_3 - 2R_1$
- Replace $R_4$ by $R_4 + 3R_1$
- Scale $R_2$ by $1/2$
- Scale $R_4$ by $1/3$
- Replace $R_5$ by $R_5 - 6R_4$.

If $A_e$ is given by

$$
A_e = \begin{pmatrix}
1 & 3 & \ast \\
5 & 0 & -2 \\
\ast & \ast & \ast \\
\end{pmatrix},
$$

compute $\det(A)$.

b) Explain why your answer from the previous part of the question does not depend on the particular sequence of row operations, i.e. why computing the determinant of a matrix using row operations in this way is well-defined.
Question 7. Suppose $A \in \mathcal{L}(\mathbb{R}^3)$ is given by the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

with respect to the standard basis.

a) Find the characteristic polynomial, $p_A(x)$, and find the eigenvalues of $A$.

b) For each eigenvalue, find the corresponding eigenspace.

c) Is $A$ diagonalizable? If it is, write its diagonalized version. If it is not, explain why.
Question 8. Let $V$ be a finite-dimensional vector space and suppose $P \in \mathcal{L}(V)$ satisfies $P^2 = P$.

a) Find all possible eigenvalues of $P$.

b) Prove that $P$ is diagonalizable and determine the diagonal basis for $P$ in terms of the bases of its fundamental subspaces.