Math 162 (51) - Final Exam
Winter Quarter 2017
Thursday, March 16, 2017

Instructions:

• Read each problem carefully.
• Write legibly.
• Show all your work on these sheets. Feel free to use the opposite side.
• This exam has 9 pages, and 8 problems. Please make sure that all pages are included.
• Each problem is worth 10 points.
• You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
• Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
Question 1. Let $f : [a,b] \to \mathbb{R}$ be continuous. Prove that $f$ is integrable on $[a,b]$. 
Question 2. Let \( f, g \) be continuous, non-negative functions on \([a,b]\). Suppose that for some constant \( C \) we have

\[
f(x) \leq C + \int_a^x f(t)^\alpha g(t) dt,
\]

for some \( 0 < \alpha < 1 \). Prove that

\[
f(x) \leq \left( (1 - \alpha) \int_a^x g(t) dt + C^{1-\alpha} \right)^{\frac{1}{1-\alpha}}
\]

and justify your work.
Question 3. Prove that

\[ \int_1^n \log x \, dx = n \log n - n + 1. \]

Use this to show that

\[ \log(n!) \geq n \log n - n + 1 \]

and deduce a lower bound for \( n! \) from this inequality.
Question 4. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function with $\int_{-\infty}^{\infty} \phi(t) \, dt = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{h \to 0} \int_{-\infty}^{\infty} \frac{1}{h} \phi \left( \frac{y}{h} \right) f(x - y) \, dy = f(x)$$

for all $x \in \mathbb{R}$. Do not quote the similar homework problem we had. Hint: consider the expression $\lim_{h \to 0} \int_{-\infty}^{\infty} \frac{1}{h} \phi \left( \frac{y}{h} \right) f(x) \, dy$. 
Question 5.  (a) (7 points) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$  \hspace{1cm} (1)

Determine $P_{n,0,f}(x)$ for $n \in \mathbb{N}$. Note: you may not assume the formula for the Taylor polynomial of $\sin x$.

(b) (3 points) What degree Taylor polynomial is needed to compute $\frac{\sin x}{x}$ for $x = 10^{-3}$ to an accuracy of $10^{-6}$. You may assume

$$R_{n,0,f} = \int_0^x \frac{f^{(n+1)}}{n!}(t)(x-t)^n dt$$

or any other form of the remainder.
Question 6. Consider the sequence \( \{a_n\}_{n=1}^{\infty} \) defined by

\[
a_1 = 1 \\
\]

\[
a_{n+1} = 5 + \frac{a_n}{4}.
\]

Prove that this sequence converges and compute its limit.
Question 7. Prove using the Bolzano-Weierstrass theorem that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.
Question 8. Suppose that \( \sum_{n=1}^{\infty} a_n \) converges (but not necessarily absolutely) and \( \{b_n\}_{n=1}^{\infty} \) is monotonic and bounded. Prove that \( \sum_{n=1}^{\infty} a_n b_n \) converges. Hint: Let \( s_n = \sum_{k=1}^{n} a_k \) and use the identity \( a_n = s_n - s_{n-1} \).