**Instructions:**

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets. Feel free to use the opposite side.
- This exam has 9 pages, and 8 problems. Please make sure that all pages are included.
- Each problem is worth 10 points.
- You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
- Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
Question 1. Let $a, b, c, d$ be numbers and suppose that $a < b$ and $c < d$. Prove using only (P1) - (P12) (and the definition of $<$) that $a + c < b + d$. 
Question 2. Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a function such that for every \( n \in \mathbb{N} \setminus \{0\} \), 
\[ |f(x)| \leq \frac{1}{n} \] except at finitely many points. Prove that \( \lim_{x \to a} f(x) = 0 \) for every \( a \in [0, 1] \).
Question 3. Consider a set $A \subseteq \mathbb{R}$, $A \neq \emptyset$ which is bounded below. Let

$$B = \{ \alpha : \alpha \text{ is a lower bound for } A \}$$

Show that $\sup B$ exists and $\sup B = \inf A$. 
Question 4. Prove using the $\varepsilon$-$\delta$ definition of uniform continuity that the function $f(x) = x^3$ is not uniformly continuous on $(0, \infty)$. 
Question 5. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function with \( \lim_{x \to \pm \infty} f(x) = 0 \) and \( f(x) > 0 \) for all \( x \in \mathbb{R} \). Prove there exists \( y \in \mathbb{R} \) such that \( f(x) \leq f(y) \) for all \( x \in \mathbb{R} \).
Question 6. (a) (7 points) Let \( f : \mathbb{R} \to \mathbb{R} \), with
\[
f(x) = \begin{cases} 
x^n & x \geq 0 \\
0 & x < 0.
\end{cases}
\]
Derive a formula for \( f^{(k)}(x) \) for \( 1 \leq k \leq n - 1 \) and all \( x \in \mathbb{R} \) and prove your claim by induction using the definition of derivatives.

(b) (3 points) Use this formula to show that \( f^{(k)}(0) \) exists for \( 1 \leq k \leq n - 1 \) but \( f^{(n)}(0) \) does not.
Question 7.  
(a) (8 points) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose that $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$. Prove that for any $a \in \mathbb{R}$, $f(x) - f(a) \leq g(x) - g(a)$ for all $x \geq a$.

(b) (2 points) Conclude that under the hypotheses of part (a), if $f(a) = g(a)$ for some $a \in \mathbb{R}$, then $f(x) \leq g(x)$ for all $x \geq a$. 
Question 8. (a) (6 points) Let $f : [a, b] \to \mathbb{R}$ be differentiable and convex. Prove that there exists $c \in [a, b]$ such that $f$ is decreasing on $[a, c]$ and increasing on $[c, b]$. What is the significance of the point $c$? Is it unique?

(b) (4 points) Under the hypotheses of part (a), let $g$ be the function given by restricting $f$ to $[c, b]$, that is $g : [c, b] \to \mathbb{R}$ is given by

$$g(x) = f(x), \quad x \in [c, b].$$

Prove that $g$ is invertible (you must also determine the domain and codomain of its inverse). What can you say about $(g^{-1})'$?