Math 161 (51) - Final Exam
Autumn Quarter 2016
Thursday, December 8, 2016

Instructions:

• Read each problem carefully.
• Write legibly.
• Show all your work on these sheets. Feel free to use the opposite side.
• This exam has 10 pages, and 9 problems. Please make sure that all pages are included.
• Each problem is worth 10 points. The true / false questions are worth 2 points each for a total of 10 points.
• You may not use books, notes, calculators, etc. Cite theorems from class or from the texts as appropriate.
• Proofs should be presented clearly (in the style used in lectures) and explained using complete English sentences.

Good luck!
Question 1. (True/False.)

1. The sum of two irrational numbers is always irrational.

2. The function $f : [0, \pi/2] \rightarrow \mathbb{R}$, $f(x) = \cos(x)$ is one-one.

3. Every subset of $\mathbb{R}$ has a least upper bound.

4. A differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ with bounded derivative is uniformly continuous.

5. Every continuous function $f : (0, 1) \rightarrow \mathbb{R}$ is bounded.
Question 2. Let \(a, b, c, d\) be numbers and suppose that \(a < b\) and \(c < d\). Prove using only (P1) - (P12) (and the definition of \(<\)) that \(a + c < b + d\).
Question 3. Prove that \( n! \geq n^2 \) for all \( n \geq 4 \). Recall: \( n! = n \cdot (n - 1) \cdots 2 \cdot 1 \).
Question 4. 1. (8 points) Prove the squeeze theorem: Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be functions and suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \in \mathbb{R}$.
Suppose that $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$. Then $\lim_{x \to a} f(x) = L$.

2. (2 points) How can you weaken the hypotheses of this theorem while maintaining the same conclusion?
Question 5. Prove using the $\varepsilon$-$\delta$ definition of uniform continuity that the function $f(x) = x^3$ is not uniformly continuous on $(0, \infty)$. 
Question 6. Let

\[ A = \left\{ 1 - \frac{1}{2^n} : n \in \mathbb{N} \right\}. \]

Prove that \( \sup A \) exists and determine its value (with proof).
Question 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = 0$ and $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that there exists $y \in \mathbb{R}$ such that $f(x) \leq f(y)$ for all $x \in \mathbb{R}$. 
Question 8. Let $f : \mathbb{R} \to \mathbb{R}$, with

$$f(x) = \begin{cases} x^n & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Derive a formula (with proof) for $f^{(k)}(x)$ for $1 \leq k \leq n - 1$ and all $x \in \mathbb{R}$ and prove that $f^{(k)}(0)$ exists for $1 \leq k \leq n - 1$ but $f^{(n)}(0)$ does not.
Question 9. Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be continuous. Suppose that for all $x \in (-a, a)$, $f'(x)$ exists and $f'(x) \leq 1$. If $f(a) = a$ and $f(-a) = -a$, prove that $f(x) = x$ for all $x \in [-a, a]$. 