Problem 5.23 Solution

- **Problem 5.23** Here, it is easiest to prove the *contrapositive* of the statements involved, that is, instead of proving that for statements $A, B$, $A \Rightarrow B$ we instead show that $(\neg B) \Rightarrow (\neg A)$. [Try to convince yourself that these implications are indeed equivalent by formulating a proof by contradiction for the first implication].

**Solution.** We are trying to show that for a given function $f(x)$,

whenever $\lim_{x \to 0} g(x)$ does not exist, $\lim_{x \to 0} f(x)g(x)$ does not exist $\iff \lim_{x \to 0} f(x)$ exists and is $\neq 0$.

We are considering the limits in the first part of the statement in the restricted sense, that is, only finite limits. Note that the condition $\lim_{x \to 0} f(x) \neq 0$ is necessary, since in the case that $\lim_{x \to 0} f(x) = 0$, we already have an example where this does not hold, namely the functions $\sin(1/x)$ and $x$.

The first two parts of the problem prove the “only if” part of this statement, that is, **if** $\lim_{x \to 0} f(x)$ exists and is non-zero **then** for any $g$ such that $\lim_{x \to 0} g(x)$ does not exist, $\lim_{x \to 0} f(x)g(x)$ does not exist. We prove this by showing the contrapositive: **if** $\lim_{x \to 0} f(x)$ exists, **then** when $\lim_{x \to 0} f(x)g(x)$ exists, $\lim_{x \to 0} g(x)$ exists as well.

(a) Suppose that $\lim_{x \to 0} f(x)$ exists and is $\neq 0$. Then if $\lim_{x \to 0} g(x)$ does not exist, $\lim_{x \to 0} f(x)g(x)$ does not exist.

**Proof.** Since $\lim_{x \to 0} f(x) \neq 0$, there exists $\delta_0 > 0$ such that if $|x| < \delta_0$, $f(x) \neq 0$, and thus if $\lim_{x \to 0} f(x)g(x)$ exists, we have $\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} f(x)g(x)/f(x)$.

(b) Suppose that $\lim_{x \to 0} |f(x)| = \infty$. Then if $\lim_{x \to 0} g(x)$ does not exist, $\lim_{x \to 0} f(x)g(x)$ does not exist.

**Proof.** If $\lim_{x \to 0} f(x)g(x) = \ell < \infty$, then for all $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$0 < |x| < \delta \implies |f(x)g(x) - \ell| < \varepsilon,$$

but for any $M \in \mathbb{N}$ there exists $\delta_1$ such that $|x| < \delta_1$ implies $|f(x)| > M$, hence for such $x$ we have

$$|f(x)g(x) - \ell| > |f(x)||g(x)| - |\ell| > M|g(x)| - |\ell|$$

so $g(x)$ cannot be bounded below, and hence $\lim_{x \to 0} g(x) = 0$.

(c) Finally, we prove the “if” portion of the statement, again by proving the contrapositive: if $\lim_{x \to 0} f(x)$ does not exist or is zero then it is not the case that when $\lim_{x \to 0} g(x)$ does not exist, $\lim_{x \to 0} f(x)g(x)$ does not exist either. We can rephrase this as: if $\lim_{x \to 0} f(x)$ does not exist (neither finite nor infinite), then there exists some $g(x)$ such that $\lim_{x \to 0} g(x)$ does not exist but $\lim_{x \to 0} f(x)g(x)$ does exist.

**Proof.** We have already addressed the case when $\lim_{x \to 0} f(x) = 0$ with the example above. Now we consider two cases: when $f(x)$ is bounded away from 0 and when it is not. Suppose first that $|f(x)| > \varepsilon$ for $x$ sufficiently close to 0. Then we can take $g = \frac{1}{f}$, and $\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} 1 = 1$. 
When $f(x)$ is not bounded below, we consider a sequence $x_n \to 0$ so that $f(x_n) < \frac{1}{n}$. Define $g(x) = 0$ for all $x \neq x_n$ for any $n$, and $g(x_n) = 1$. Note that $\lim_{x \to 0} g(x)$ does not exist since we can always find $|x_n| < \delta$ with $g(x_n) = 1$ and $|x| < \delta$ with $g(x) = 0$. However,

$$f(x)g(x) = \begin{cases} 
0 & x \neq x_n \text{ for all } n \in \mathbb{N} \\
\frac{1}{n} & x = x_n,
\end{cases}$$

and hence $\lim_{x \to 0} f(x)g(x) = 0$. 