CONVOLUTION ALGEBRAS AND APPLICATIONS TO REPRESENTATION THEORY

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1. Introduction

This exposition is mainly based on [CG] and [Gi]; we use general results about convolution algebras on Borel-Moore homology and other related constructions to study the representations of $\mathcal{U}(\mathfrak{sl}_n)$ and other interesting objects like the group algebra of a Weyl group and affine Hecke algebras.

2. Convolution in Borel-Moore Homology

We start by briefly giving two equivalent definitions of Borel-Moore homology (details can be found in [BM], [Br]); all the spaces that we are going to deal with will be complex algebraic varieties and we are always going to consider all homology and cohomology to be taken with complex coefficients.

Definition 2.1. (1) Let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of X. Define $H_*^{BM}(X) = H_*(\hat{X}, \infty)$ where H_* is the ordinary relative homology for the pair $(\hat{X}, \infty)$.

(2)Poincaré duality: if M is smooth, $\dim_{\mathbb{R}} M = m$, let $X \subset M$ be closed with a closed neighborhood $U \subset M$, such that X is a proper deformation retract of U. Then there is a canonical isomorphism $H_i^{BM}(X) \simeq H^{m-i}(M, M \setminus X)$.

From now on we will denote H_*^{BM} just by H_*.

Properties.

Proper Pushforward. If $f : X \to Y$ is proper, then we have the direct image map

$$f_* : H_*(X) \to H_*(Y).$$

Restriction. If $U \subset X$ is open, then there is a natural restriction morphism $H_*(X) \to H_*(U)$.

Fundamental Class. If X is smooth (not necessarily compact), $\dim_{\mathbb{R}} X = m$, then there is a well-defined fundamental class, $[X] \in H_m(X)$. If X is irreducible but not smooth, the restriction to the Zariski open dense subset of regular points is an isomorphism. $H_m(X) \xrightarrow{\sim} H_m(X^{reg})$ so we can define $[X]$ to be the preimage of $[X^{reg}]$. If X is any variety with irreducible components $X_1, X_2, \dots, X_n$, then we set $[X] = \sum_i [X_i]$ a non-homogenous class.

Proposition 2.2. *Let X be a complex variety of complex dimension n and let $X_1, \dots, X_m$ be the n-dimensional irreducible components of X. Then the classes $[X_1], \dots, [X_m]$ form a basis for the vector space $H_{top}(X) = H_{2n}(X)$.*

Intersection Pairing. Let M be smooth, $\dim_{\mathbb{R}} M = m$ and let $Z, \tilde{Z} \subset M$, closed with the same condition as in definition 2.1 (4). Then we have the cup product on relative cohomology:

$$\cup : H^{m-i}(M, M \setminus Z) \times H^{m-j}(M, M \setminus \tilde{Z}) \to H^{2m-i-j}(M, (M \setminus Z) \cup (M \setminus \tilde{Z}))$$

which by Poincaré duality becomes the intersection product

$$\cap : H_i(Z) \times H_j(\tilde{Z}) \to H_{i+j-m}(Z \cap \tilde{Z}). \tag{1}$$

2.1. **Convolution.** Let M_1, M_2, M_3 be smooth complex varieties and let $Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_2 \times M_3$ be closed subsets. Define the set-theoretic composition

$$Z_{12} \circ Z_{23} := \{(m_1, m_3) \in M_1 \times M_3 | \quad \exists m_2 \in M_2 \text{ such that } (m_1, m_2) \in Z_{12} \text{ and } (m_2, m_3) \in Z_{23}\}.$$

Let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection and let us assume from now on that

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3 \quad \text{is proper.} \tag{2}$$

Remark that $Z_{12} \circ Z_{23}$ equals the image of the map in (2), so in particular it is a closed subset of $M_1 \times M_3$.

Definition 2.3. Let $d = \dim_{\mathbb{R}} M_2$, we define a *convolution* in Borel-Moore homology in this way

$$H_i(Z_{12}) \times H_j(Z_{23}) \to H_{i+j-d}(Z_{12} \circ Z_{23}) \tag{3}$$

$$(c_{12}, c_{23}) \mapsto c_{12} * c_{23} := (p_{13})_*((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23}))$$

where $c_{12} \boxtimes [M_3]$, $[M_1] \boxtimes c_{23}$ are given by the Künneth formula and $\cap$ is the intersection defined in (1).

Associativity of Convolution. If we are given a fourth smooth variety M_4 and a closed subset $Z_{34} \subset M_3 \times M_4$ we have

$$(c_{12} * c_{23}) * c_{34} = c_{12} * (c_{23} * c_{34}) \tag{4}$$

where $c_{ij} \in H_*(Z_{ij})$.

Let us now give a result that allows us to perform computations of convolution. Retaining the previous notations, assume that Z_{12} and Z_{23} are smooth and let $Z_{13} = Z_{12} \circ Z_{23}$. Also let $\mathrm{pr}_{ij} : T^*(M_1 \times M_2 \times M_3) \to T^*(M_i \times M_j)$ the map induced on cotangent bundles by p_{ij}. We relate the convolution product of Z_{ij} to the convolution of the conormal bundles $X_{ij} := T^*_{Z_{ij}}(M_i \times M_j)$.

Theorem 2.4. *Suppose that the intersection of $p_{12}^{-1}(Z_{12})$ and $p_{23}^{-1}(Z_{23})$ is transverse and that the map p_{13} in (2) is a smooth locally trivial oriented fibration with smooth base Z_{13} and smooth and compact fiber F. Then*

(i) *We have a set-theoretic equality $X_{12} \circ X_{23} = X_{13}$;*
(ii) $\mathrm{pr}_{13} : \mathrm{pr}_{12}^{-1}(X_{12}) \cap \mathrm{pr}_{23}^{-1}(X_{23}) \to X_{13}$ *is a smooth locally trivial oriented fibration with fiber F;*
(iii) *In $H_*(X_{13})$ we have the equation $[X_{12}] * [X_{23}] = \chi(F)[X_{13}]$ where $\chi(F)$ is the Euler characteristic.*

2.2. **Convolution in Equivariant K-theory.** Given a complex linear algebraic group G and an algebraic G-variety X, we can consider the abelian category $\mathrm{Coh}^G(X)$ of G-equivariant coherent sheaves on X. We let $K^G(X)$ be the Grothendieck group of $\mathrm{Coh}^G(X)$, which has a natural $\mathbf{R}(G)$-module structure, where $\mathbf{R}(G) = K^G(pt)$ is the representation ring of G. Remember that there are proper direct image morphisms and restriction morphisms for equivariant K-theory, as in Borel-Moore homology. Now, in the convolution setup, let M_1, M_2, M_3 be smooth G varieties. We have the induced G-action on the cartesian product that makes the projections G-equivariant. So, if $Z_{12} \subset M_1 \times M_2$ and $Z_{13} \subset M_2 \times M_3$ are G-stable closed subvarieties and p_{13} is proper, we have a convolution map

$$K^G(Z_{12}) \times K^G(Z_{23}) \to K^G(Z_{12} \circ Z_{23}) \tag{5}$$

$$[\mathcal{F}_{12}] * [\mathcal{F}_{23}] := (p_{13})_*(p_{12}^*[\mathcal{F}_{12}] \overset{L}{\otimes} p_{23}^*[\mathcal{F}_{23}])$$

where $\overset{L}{\otimes}$ is defined by taking finite locally free G-equivariant resolutions on the ambient smooth space $M_1 \times M_2 \times M_3$.

2.3. **The Convolution Algebra.** Let M be a smooth complex variety, let N be a (possibly singular) variety and let $\mu : M \to N$ be a proper map. In the convolution setup, put $M_1 = M_2 = M_3 = M$ and $Z = Z_{12} = Z_{23} = M \times_N M$. Explicitly we have

$$Z = \{(m_1, m_2) \in M \times M | \mu(m_1) = \mu(m_2)\}.$$

Clearly $Z \circ Z = Z$, therefore we have the convolution maps as in (3) and (5)

$$H_*(Z) \times H_*(Z) \to H_*(Z) \quad \text{and } K^G(Z) \times K^G(Z) \to K^G(Z).$$

From (4) we get immediately

Corollary 2.5. *$H_*(Z)$ has a natural structure of an associative algebra with unit and $K^G(Z)$ has a natural structure of an associative $\mathbf{R}(G)$-algebra with unit. The unit is given respectively by the fundamental class of $M_\Delta \subset Z$ and by the stucture sheaf of M_Δ.*

Now choose $x \in N$, set $M_x = \mu^{-1}(x)$. Apply the convolution construction to $M_1 = M_2 = M$, M_3 a point, $Z = Z_{12} = M \times_N M$ and $Z_{23} = M_x \subset M \times pt$. Then $Z \circ M_x = M_x$.

Corollary 2.6. *$H_*(M_x)$ has a natural structure of a left $H_*(Z)$-module under the convolution map.*

The Dimension Property. In the convolution setup, let M_1, M_2, M_3 be smooth with $\dim_{\mathbb{R}} M_i = m_i$ and let $p = \frac{m_1+m_2}{2}$, $q = \frac{m_2+m_3}{2}$ and $r = \frac{m_1+m_3}{2}$ then it is obvious from (3) that convolution induces a map

$$H_p(Z_{12}) \times H_q(Z_{23}) \to H_r(Z_{12} \circ Z_{23}). \tag{6}$$

3. Constructible Complexes, Perverse Sheaves and the Decomposition Theorem

Let X be a complex algebraic variety and let $\mathcal{S}h(X)$ be the abelian category of sheaves of $\mathbb{C}$-vector spaces over X. We consider the bounded derived category $D^b(\mathcal{S}h(X))$, where we can represent objects by bounded complexes of sheaves. A sheaf $\mathcal{F}$ on X is said to be *constructible* if there is a finite algebraic stratification $X = \sqcup_\alpha X_\alpha$, such that for all α, each stratum X_α is a locally closed smooth connected algebraic subvariety of X and the restriction of $\mathcal{F}$ to X_α is a *local system* (a locally constant sheaf of finite dimensional vector spaces). An object $A \in D^b(\mathcal{S}h(X))$ is a *constructible complex* if all the cohomology sheaves $\mathcal{H}^i(A)$ are constructible. Let $D^b(X)$ be the full subcategory of $D^b(\mathcal{S}h(X))$ formed by constructible complexes.

Yoneda Product. Let X be a variety, $A_1, A_2, A_3 \in D^b(X)$. For any $p, q \in \mathbb{Z}$ the composition of morphisms gives a bilinear product

$$\mathrm{Hom}_{D^b(X)}(A_1, A_2[p]) \times \mathrm{Hom}_{D^b(X)}(A_2[p], A_3[p+q]) \to \mathrm{Hom}_{D^b(X)}(A_1, A_3[p+q]).$$

Using that $\mathrm{Hom}_{D^b(X)}(A_2[p], A_3[p+q]) = \mathrm{Hom}_{D^b(X)}(A_2, A_3[q]) = \mathrm{Ext}^q_{D^b(X)}(A_2, A_3)$, we can rewrite the composition above as a bilinear product on Ext groups called the *Yoneda product*

$$\mathrm{Ext}^p_{D^b(X)}(A_1, A_2) \otimes \mathrm{Ext}^q_{D^b(X)}(A_2, A_3) \to \mathrm{Ext}^{p+q}_{D^b(X)}(A_1, A_3). \tag{7}$$

Perverse Sheaves. We will just give some terminology in order to state the Decomposition Theorem; for a detailed treatment of perverse sheaves see [BBD].

Objects in $D^b(X)$ that satisfy a certain condition are called *perverse sheaves*, for example, for X a smooth variety with irreducible components X_i, the complex $\mathcal{C}_X$ on X, defined by the equality

$$\mathcal{C}_X|_{X_i} = \mathbb{C}_{X_i}[\dim_{\mathbb{C}} X_i]$$

is called the *constant perverse sheaf* on X.

Another very important example are *intersection cohomology complexes* $IC(Y, \mathcal{L})$ (or sometimes just $IC_{\mathcal{L}}$) which are defined uniquely by $Y \subset X$ a smooth locally closed subvariety and a local system $\mathcal{L}$ on Y.

Theorem 3.1. [BBD]*(i) The full subcategory of $D^b(X)$ whose objects are perverse sheaves on X is an abelian category which we will denote by* $\mathrm{Perv}(X)$. *(ii) The simple objects of* $\mathrm{Perv}(X)$ *are the intersection complexes $IC(Y, \mathcal{L})$ as $\mathcal{L}$ runs through the irreducible local systems on various smooth locally closed subvarieties $Y \subset X$.*

Corollary 3.2. *There are no negative degree* Ext*-groups between perverse sheaves, in particular*

$$\mathrm{Ext}^k_{D^b(X)}(IC_\varphi, IC_\psi) = 0 \qquad \text{for all } k < 0,$$

Also, if φ, ψ are irreducible local systems, we have

$$\mathrm{Ext}^0_{D^b(X)}(IC_\varphi, IC_\psi) = \mathrm{Hom}_{D^b(X)}(IC_\varphi, IC_\psi) = \mathrm{Hom}_{\mathrm{Perv}(X)}(IC_\varphi, IC_\psi) = \mathbb{C} \cdot \delta_{\varphi,\psi}.$$

Recall that a morphism $\mu : M \to N$ is called *projective* if it can be factored as a composition of a closed embedding $M \hookrightarrow \mathbb{P}^n \times N$ and the projection $\mathbb{P}^n \times N \to N$. Any proper algebraic map between quasi-projective varieties is known to be projective.

Theorem 3.3 (Decomposition Theorem)**.** [BBD] *Let $\mu : M \to N$ a projective morphism and $X \subset M$ a smooth locally closed subvariety. Then we have a finite direct sum decomposition in $D^b(N)$*

$$\mu_* IC(X, \mathbb{C}_X) = \bigoplus_{(i,Y,\chi)} L_{Y,\chi}(i) \otimes IC(Y, \chi)[i],$$

where Y are locally closed subvarieties of N, χ is an irreducible local system on Y, $i \in \mathbb{Z}$ and $L_{Y,\chi}(i)$ are some finite dimensional vector spaces.

Remark 3.4. Now suppose that, in the setting of the theorem, M is smooth and $N = \sqcup N_\alpha$ an algebraic stratification such that, for every α, $\mu : \mu^{-1}(N_\alpha) \to N_\alpha$ is a locally trivial topological fibration (such a stratification always exists). Applying the Decomposition Theorem to $\mu_* \mathcal{C}_M$ we see that all complexes on the RHS have locally constant cohomology sheaves along each stratum N_α, thus the decomposition becomes

$$\mu_* \mathcal{C}_M = \bigoplus_{k \in \mathbb{Z}, \varphi = (N_\alpha, \chi_\alpha)} L_\varphi(k) \otimes IC_\varphi[k], \tag{8}$$

where $IC_\varphi = IC(N_\alpha, \chi_\alpha)$.

4. Sheaf-theoretic Analysis of the Convolution Algebra and Semi-small Maps

In this section, we see how we can use the Decomposition Theorem to get information about the convolution algebra. Let us consider the setting of the convolution algebra, M is a smooth variety, N is any variety and $\mu : M \to N$ is proper map, then we have $Z = M \times_N M$.

Proposition 4.1. *There exists a (not necessarily grading preserving) algebra isomorphism*

$$H_\bullet(Z) \simeq \mathrm{Ext}^\bullet_{D^b(N)}(\mu_* \mathcal{C}_M, \mu_* \mathcal{C}_M), \tag{9}$$

where the product on the RHS is the Yoneda product as in (7)*.*

Now suppose that the conditions in Remark 3.4 are satisfied, then we can apply (8) to (9) to get

$$\begin{aligned} H_\bullet(Z) &\simeq \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^k_{D^b(N)}(\mu_* \mathcal{C}_M, \mu_* \mathcal{C}_M) \\ &= \bigoplus_{i,j,k \in \mathbb{Z}, \varphi, \psi} \mathrm{Hom}_{\mathbb{C}}(L_\varphi(i), L_\psi(j)) \otimes \mathrm{Ext}^{k+j-i}_{D^b(N)}(IC_\varphi, IC_\psi) \end{aligned}$$

now let $L_\varphi := \oplus_{i \in \mathbb{Z}} L_\varphi(i)$ and use Corollary 3.2 to simplify the terms to get

$$\begin{aligned} H_\bullet(Z) &\simeq \bigoplus_{\varphi,\psi} \left(\mathrm{Hom}_{\mathbb{C}}(L_\varphi, L_\psi) \otimes \mathrm{Hom}_{D^b(N)}(IC_\varphi, IC_\psi)\right) \bigoplus_{k>0,\varphi,\psi} \left(\mathrm{Hom}(L_\varphi, L_\psi) \otimes \mathrm{Ext}^k_{D^b(N)}(IC_\varphi, IC_\psi)\right) \\ &= \left(\bigoplus_{\varphi} \mathrm{End}_{\mathbb{C}}(L_\varphi)\right) \oplus \left(\bigoplus_{k>0,\varphi,\psi} \mathrm{Hom}(L_\varphi, L_\psi) \otimes \mathrm{Ext}^k_{D^b(N)}(IC_\varphi, IC_\psi)\right) \end{aligned}$$

The first sum in this expression is a semisimple subalgebra, while the second sum is concentrated in degrees $k > 0$, hence is a nilpotent ideal $H_\bullet(Z)_+ \subset H_\bullet(Z)$. Now consider

$$H_\bullet(Z) \twoheadrightarrow H_\bullet(Z)/H_\bullet(Z)_+ \simeq \bigoplus_\varphi \mathrm{End}(L_\varphi) \twoheadrightarrow L_\psi;$$

this tells us that $H_\bullet(Z)_+$ is the radical of $H_\bullet(Z)$ and we have that for every ψ the projection map on the ψ-summand gives us an irreducible representation of $H_\bullet(Z)$ and every simple $H_\bullet(Z)$-module is of this form. This gives us the following result.

Theorem 4.2. *The non-zero members of the collection $\{L_\varphi\}$ arising from* (8) *form a complete set of isomorphism classes of simple $H_\bullet(Z)$-modules.*

4.1. **Semi-small Maps.** As before, we have M smooth variety, N a variety $\mu : M \to N$ a proper (hence projective) map. Let $M_1, \dots, M_r$ be the connected components of M and let $N = \sqcup_\alpha N_\alpha$ be an algebraic stratification. Given $x \in N_\alpha$ let $M_x = \mu^{-1}(x)$.

The following notion is introduced in [GM].

Definition 4.3. The morphism μ is *semi-small* with respect to the stratification $N = \sqcup N_\alpha$ if, for every component M_k, we have

$$\dim_{\mathbb{C}} N_\alpha + 2\dim_{\mathbb{C}}(M_x \cap M_k) \le \dim_{\mathbb{C}} M_k \qquad \text{for all } \alpha \text{ such that } N_\alpha \subset \mu(M_k). \tag{10}$$

If we always have equality in (10), then we say that μ is *stricly semi-small.*

In the case of semi-small maps we get nicer results from the Decomposition Theorem. Let $Z_{ij} := M_i \times_N M_j$ and set

$$H(Z) = \oplus_{ij} H_{m_i+m_j}(Z_{ij}) \qquad \text{where } m_i = \dim_{\mathbb{C}} M_i,$$

then by (6) $H(Z)$ is a subalgebra of $H_\bullet(Z)$. Now, for $x \in N_\alpha$, let

$$H(M_x) = \oplus_k H_{2\dim_{\mathbb{C}}(M_x \cap M_k)}(M_x \cap M_k).$$

Theorem 4.4. *(i)If μ is semi-small, then $\mu_*\mathcal{C}_M$ is perverse and we have a decomposition without shifts*

$$\mu_*\mathcal{C}_M = \bigoplus_{\varphi=(N_\varphi,\chi_\varphi)} L_\varphi \otimes IC_\varphi. \tag{11}$$

Furthermore, we have an algebra isomorphism

$$H(Z) \simeq \mathrm{Hom}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M) = \bigoplus_\varphi \mathrm{End}_{\mathbb{C}} L_\varphi.$$

(ii)For any stratum N_α, the family of spaces $\{H(M_x), x \in N_\alpha\}$ forms a local system on N_α. If L_φ is the multiplicity space in (11) *such that $x \in N_\varphi$ and χ_φ is the representation of $\pi_1(N_\varphi, x)$ associated with φ, then*

$$L_\varphi = H(M_x)_\varphi = \mathrm{Hom}_{\pi_1(N_\varphi,x)}(H(M_x), \chi_\varphi).$$

Here we used the fact that, for a connected, locally simply connected space N and a point $x \in N$, there is an equivalence of categories between local systems on N and representations of $\pi_1(N, x)$, sending a local system to its fiber at x, which is naturally a $\pi_1(N, x)$-module by means of the monodromy action.

5. Springer Theory for $\mathcal{U}(\mathfrak{sl}_n)$

Our goal in this section is to apply the general machinery we have seen so far, to the special case of constructing representations of $\mathfrak{sl}_n$. We fix the corresponding $n \ge 1$ and an other integer $d \ge 1$ with no relation to n. An n-step *partial flag* F in $\mathbb{C}^d$ is a sequence of subspaces $0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^d$ where the inclusions are not necessarily proper. The set $\mathcal{F}$ of all partial flags is a smooth compact manifold with connected components corresponding to partitions $\mathbf{d} = (d_1 + d_2 + \cdots + d_n = d)$ where $d_i \in \mathbb{Z}_{\ge 0}$, so

$$\mathcal{F} = \sqcup_{\mathbf{d}} \mathcal{F}_{\mathbf{d}} \quad \text{where} \quad \mathcal{F}_{\mathbf{d}} = \{F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^d) \in \mathcal{F} \mid \dim(F_i/F_{i-1}) = d_i\}.$$

Now, let us define $N := \{x \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^d) | x^n = 0\}$, and let $M := \{(x, F) \in N \times \mathcal{F} | x(F_i) \subset F_{i-1}, i = 1, \ldots, n\}$. The first and second projections give rise to a natural diagram

$$N \xleftarrow{\mu} M \xrightarrow{\pi} \mathcal{F}$$

The natural action of $\mathrm{GL}_d(\mathbb{C})$ on $\mathbb{C}^d$ gives rise to $\mathrm{GL}_d(\mathbb{C})$-actions on $\mathcal{F}, N$ and M by conjugation and these actions clearly commute with the projections.

Proposition 5.1. *There is a natural* $\mathrm{GL}_d(\mathbb{C})$*-equivariant vector bundle isomorphism* $M \simeq T^*\mathcal{F}$ *making the map* π *into the canonical projection* $T^*\mathcal{F} \to \mathcal{F}$.

The decomposition of $\mathcal{F}$ into connected components gives a decomposition of $M = \sqcup_{\mathbf{d}} M_{\mathbf{d}}$ with $M_{\mathbf{d}} = T^*\mathcal{F}_{\mathbf{d}}$. The variety N is naturally stratified by $\mathrm{GL}_d(\mathbb{C})$-conjugacy classes $N = \sqcup_\alpha N_\alpha$. For a point $x \in N$ we have $\mu^{-1}(x) = \{(x, F) | x(F_i) \subset F_{i-1}\}$ which can be identified with a subvariety $\mathcal{F}_x$ of $\mathcal{F}$.

Lemma 5.2. [Sp]*For any* $x \in N$ *and a* n*-step partition* $\mathbf{d}$, $\mathcal{F}_x \cap M_{\mathbf{d}}$ *is a connected variety of pure dimension and*

$$\dim \mathbb{O}_x + 2\dim(\mathcal{F}_x \cap M_{\mathbf{d}}) = 2\dim \mathcal{F}_{\mathbf{d}} \tag{12}$$

where $\mathbb{O}_x$ *denotes the* $\mathrm{GL}_d(\mathbb{C})$*-orbit of* x.

As before, set $Z = M \times_N M$ and consider the convolution algebra $H_*(Z)$. By (12), μ is strictly semi-small therefore, by Theorem 4.4, the subspace $H(Z) \subset H_*(Z)$ spanned by the fundamental classes of the irreducible components of Z is a semisimple subalgebra. We know that $H_*(\mathcal{F}_x)$ is a $H_*(Z)$-module via convolution, using (6) we get that the subspace $H(\mathcal{F}_x) \subset H_*(\mathcal{F}_x)$, spanned by the fundamental classes of the irreducible components of $\mathcal{F}_x$ is stable under $H(Z)$-action. Following Theorem 4.4 (ii) we should decompose $H(\mathcal{F}_x)$ into isotypical components with respect to the monodromy action and the multiplicity spaces will be the irreducible modules over $H(Z)$.

Lemma 5.3. *The monodromy action on* $H(\mathcal{F}_x)$ *is trivial for any* $x \in N$.

Corollary 5.4. *If* $x, y \in N$ *are* $\mathrm{GL}_d(\mathbb{C})$*-conjugate, then* $H(\mathcal{F}_x)$ *and* $H(\mathcal{F}_y)$ *are isomorphic as* $H(Z)$*-modules. The spaces* $\{H(\mathcal{F}_x)\}$ *where* x *runs over the representatives of* $\mathrm{GL}_d(\mathbb{C})$*-conjugacy classes in* N*, form a complete collection of irreducible* $H(Z)$*-modules.*

Theorem 5.5 (Geometric Construction of $\mathcal{U}(\mathfrak{sl}_n)$)**.** *There is a natural surjective algebra morphism*

$$\mathcal{U}(\mathfrak{sl}_n) \twoheadrightarrow H(Z).$$

Sketch of the proof. First of all remark that $Z = M \times_N M \subset M \times M = T^*\mathcal{F} \times T^*\mathcal{F} \simeq T^*(\mathcal{F} \times \mathcal{F})$, and it can be shown that Z is the union of the conormal bundles to all $\mathrm{GL}_d(\mathbb{C})$-orbits in $\mathcal{F} \times \mathcal{F}$. Now, to define a map $\Theta : \mathcal{U}(\mathfrak{sl}_n) \to H(Z)$, we will show where to send the Chevalley generators $\{e_\alpha, f_\alpha, h_\alpha | \alpha = 1, \ldots, n\}$. Given a partition $\mathbf{d} = (d_1 + \cdots + d_n = d)$ we can consider (when it makes sense) the partitions

$$\mathbf{d}_\alpha^+ = d_1 + \cdots + d_{\alpha-1} + (d_\alpha + 1) + (d_{\alpha+1} - 1) + \cdots + d_n,$$
$$\mathbf{d}_\alpha^- = d_1 + \cdots + d_{\alpha-1} + (d_\alpha - 1) + (d_{\alpha+1} + 1) + \cdots + d_n.$$

Now we can define, for $\mathbf{d}$ such that $\mathbf{d}_\alpha^+$ makes sense (respectively when $\mathbf{d}_\alpha^-$ makes sense) the subvarieties

$$Y_{\mathbf{d}_\alpha^+, \mathbf{d}} = \{(F, F') \in \mathcal{F}_{\mathbf{d}_\alpha^+} \times \mathcal{F}_{\mathbf{d}} | F'_\alpha \subset F_\alpha (\mathrm{codim} = 1); F_i = F'_i \text{ for } i \neq \alpha\}$$
$$Y_{\mathbf{d}_\alpha^-, \mathbf{d}} = \{(F, F') \in \mathcal{F}_{\mathbf{d}_\alpha^-} \times \mathcal{F}_{\mathbf{d}} | F_\alpha \subset F'_\alpha (\mathrm{codim} = 1); F_i = F'_i \text{ for } i \neq \alpha\}.$$

Also, for every partition $\mathbf{d}$ we have the diagonal subvariety $\Delta \hookrightarrow \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ and we define

$$\Theta(h_\alpha) = \sum_{\mathbf{d}} (d_\alpha - d_{\alpha+1})[T^*_\Delta(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})],$$
$$\Theta(e_\alpha) = \sum_{\mathbf{d}} [T^*_{Y_{\mathbf{d}^+_\alpha,\mathbf{d}}}(\mathcal{F}_{\mathbf{d}^+_\alpha} \times \mathcal{F}_{\mathbf{d}})],$$
$$\Theta(f_\alpha) = \sum_{\mathbf{d}} [T^*_{Y_{\mathbf{d}^-_\alpha,\mathbf{d}}}(\mathcal{F}_{\mathbf{d}^-_\alpha} \times \mathcal{F}_{\mathbf{d}})].$$

Using Theorem 2.4 we can show that $\{\Theta(h_\alpha), \Theta(e_\alpha), \Theta(f_\alpha) | \alpha = 1, \dots, n\}$ satisfy Serre's relations, therefore we just need to show that Θ is surjective. In order to do this, first we parametrize the $\mathrm{GL}_d(\mathbb{C})$-orbits on $\mathcal{F} \times \mathcal{F}$ by the matrices

$$a_{ij}(F, F') = \dim \left(\frac{F_i \cap F'_j}{F_{i-1} \cap F'_j + F_i \cap F'_j} \right) \tag{13}$$

then we define a partial order on these matrices by $a \preceq b$ if

$$\sum_{r \le i, s \ge j} a_{rs} \le \sum_{r \le i, s \ge j} \qquad \text{for all } 1 \le i < j \le n$$
$$\sum_{r \ge i, s \le j} a_{rs} \le \sum_{r \ge i, s \le j} \qquad \text{for all } 1 \le j < i \le n.$$

By inducting on this partial order we can show that all orbits are in the image of Θ. □

Now, the finite dimensional irreducible representations of $\mathfrak{sl}_n$ are parametrized by the set of dominant weights of $\mathfrak{sl}_n$, which we can see as n-tuples of integers $d_1 \ge \dots \ge d_n$ modulo the $\mathbb{Z}$-action by simultaneous translation. Now, we want to establish a correspondance between dominant weight and $\mathrm{GL}_d(\mathbb{C})$-conjugacy classes on N which parametrize irreducible representations of $H(Z)$, by Lemma 5.4. Let $x \in N$ and let $x^0 = Id$; there are two flags associated to x:

$$F^{max}(x) = (0 = \mathrm{Ker}(x^0) \subset \mathrm{Ker}(x) \subset \dots \subset \mathrm{Ker}(x^n) = \mathbb{C}^d),$$
$$F^{min}(x) = (0 = \mathrm{Im}(x^n) \subset \dots \subset \mathrm{Im}(x) \subset \mathrm{Im}(x^0) = \mathbb{C}^d).$$

Observe that $F^{max}(x), F^{min}(x) \in \mathcal{F}_x$.

Lemma 5.6. *The partition $\mathbf{d}(x) = (d_1 + \dots + d_n = d)$, associated to $F^{max}(x)$, is a dominant weight.*

Theorem 5.7 (Springer Theorem for $\mathcal{U}(\mathfrak{sl}_n)$)**.** *For any $x \in N$ the simple $\mathfrak{sl}_n$-module $H(\mathcal{F}_x)$ has the highest weight $\mathbf{d}(x)$. In particular, every finite-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}_n$ is of the form $H(\mathcal{F}_x)$. The flags $F^{max}(x), F^{min}(x)$ are isolated points of the fiber $\mathcal{F}_x$. The fundamental classes $[F^{max}(x)], [F^{min}(x)] \in H(\mathcal{F}_x)$ are a highest weight and a lowest weight vector in $H(\mathcal{F}_x)$, respectively.*

6. Other Applications

The same general methods can be applied to study other objects that arise in Representation Theory, for example the group algebra of a Weyl group or affine Hecke algebras.

6.1. **Group Algebra of a Weyl Group.** Let us quickly sketch the setting, see [CG, chap.3] for more details: G is a complex semisimple connected Lie group, B a Borel subgroup (i.e. maximal solvable subgroup) of G, $\mathfrak{g}$ and $\mathfrak{b}$ the respective Lie algebras. Let T be a maximal torus contained in B, and $N_G(T)$ the normalizer of T in G. The quotient $W := N_G(T)/T$ is called the *Weyl group* of G. Let $\mathcal{B}$ be the set of all Borel subalgebras in $\mathfrak{g}$. Then there is a G-equivariant isomorphism of algebraic varieties $G/B \simeq \mathcal{B}$.

Now, let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{g}$ and $\tilde{\mathcal{N}} := \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} | x \in \mathfrak{b}\}$. As in the case of $\mathcal{U}(\mathfrak{sl}_n)$, we have a diagram

$$\mathcal{N} \xleftarrow{\mu} \tilde{\mathcal{N}} \xrightarrow{\pi} \mathcal{B},$$

and a G-equivariant isomorphism $\tilde{\mathcal{N}} \simeq T^*\mathcal{B}$. Then it can be shown that the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$, which is called the *Springer resolution* is proper and semi-small. Then we consider the *Steinberg variety* $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$, which is the union of the conormal bundles to the G-orbits on $\mathcal{B} \times \mathcal{B}$ (which are parametrized by W).

For $x \in \mathcal{N}$ we can identify $\mu^{-1}(x)$ with $\mathcal{B}_x \subset \mathcal{B}$, let $G(x)$ be the centralizer of x in G and $G(x)^\circ$ its identity component. Then we have that the monodromy action corresponds to the $C(x) := G(x)/G(x)^\circ$-action on $H(\mathcal{B}_x)$. So, in conclusion we have.

Theorem 6.1 (Geometric Construction of W)**.** *There is an algebra isomorphism $H(Z) \simeq \mathbb{C}[W]$. The collection $\{H(\mathcal{B}_x)_\varphi\}$, where (x, φ) runs over G-conjugacy classes of pairs $x \in \mathcal{N}, \varphi \in C(x)^\vee$ is a complete set of irreducible representation of W.*

6.2. **Affine Hecke Algebras.** In the same setting as the previous section, let $X^*(T) = \mathrm{Hom}(T, \mathbb{C}^*)$ be the weight lattice for the roots of G.

Definition 6.2. The *affine Hecke algebra* $\mathbf{H}$ of G is the $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-algebra with generators $\{T_w | w \in W\}$, $\{Y_\lambda | \lambda \in X^*(T)\}$ and relations

(1) $(T_w + 1)(T_w - \mathbf{q}) = 0$ for all $w \in W$;
(2) $T_w T_y = T_{wy}$ if $l(wy) = l(w) + l(y)$, where l is the lenght function;
(3) $Y_\lambda Y_\mu = Y_{\lambda+\mu}$;
(4) $T_{s_i} Y_\lambda - Y_{s_i(\lambda)} T_{s_i} = (1 - \mathbf{q}) \dfrac{Y_{s_i(\lambda)} - Y_\lambda}{1 - Y_{-\alpha_i}}$.

Now, there is an $A := G \times \mathbb{C}^*$-action on $\tilde{\mathcal{N}}$ defined by $(g, z) \cdot (x, \mathfrak{b}) = (z^{-1} g x g^{-1}, g \mathfrak{b} g^{-1})$, and the corresponding action on $\mathcal{N}$ such that the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is A-equivariant. Therefore we can apply the convolution construction of (5) to the K-theory of the Steinberg variety Z.

Theorem 6.3. *There is a natural algebra isomorphism $K^A(Z) \simeq \mathbf{H}$.*

We want to study the irreducible representations of $\mathbf{H}$. It is known that there is an isomorphism $\mathbf{R}(A) \simeq Z(\mathbf{H})$; now for a semisimple element $a = (s, q) \in A$ and for a character $z \in \mathbf{R}(A)$, we have $\mathbb{C}_a$ which is $\mathbb{C}$ viewed as an $\mathbf{R}(A)$-module with the action $z \cdot x = z(a)x$.

We define $\mathbf{H}_a := \mathbb{C}_a \otimes_{Z(\mathbf{H})} \mathbf{H}$ the Hecke algebra *specialized at* a. The center of $\mathbf{H}$ acts by scalars on any irreducible representation and we can see that map as the evaluation morphism of characters at some a, therefore the action of $\mathbf{H}$ factors through an action of $\mathbf{H}_a$.

Let us fix $a \in A$ from now on; a acts on $\tilde{\mathcal{N}}$, $\mathcal{N}$ and Z and we have the a-fixed point subvarieties $\tilde{\mathcal{N}}^a, \mathcal{N}^a$ and Z^a. In fact, $Z^a = \tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \tilde{\mathcal{N}}^a$ hence we have the convolution algebra as in (3).

Theorem 6.4. *If $a = (s, q) \in A$ is a semisimple element, there is an algebra isomorphism $\mathbf{H}_a \simeq H_*(Z^a)$.*

Now, as in the previous section, for $x \in \mathcal{N}^a$ identify $\mu^{-1}(x)$ with $\mathcal{B}_x^s$ and let $G(s, x)$ be the simultaneous centralizer of s and x in G, and $G(s, x)^\circ$ its identity component. Let $C(s, x) := G(s, x)/G(s, x)^\circ$, and let $L_{a,x,\chi}$ be the multiplicity space for $\chi \in C(s, x)^\vee$ acting on $H_*(\mathcal{B}_x^s)$.

Theorem 6.5. *Let $q \in \mathbb{C}^*$ not a root of unity. $L_{a,x,\chi}$ and $L_{a,x',\chi'}$ are isomorphic if and only if (x, χ) and (x', χ') are $G(s)$-conjugate to each other. The collection $\{L_{a,x,\chi}\}_{(a,x,\chi) \in \mathbf{M}}$ is a complete set of irreducible $\mathbf{H}$-modules such that $\mathbf{q}$ acts by q.*

7. Other Related Constructions

Going back to the definition of the convolution product on Borel-Moore homology, remark that if M_1, M_2 and M_3 are discrete spaces, the group $H_*(Z_{ij})$ is just the vector space $\mathbb{C}(Z_{ij})$ of $\mathbb{C}$-valued functions on Z_{ij}, therefore, we can replace the 'proper' in (2) by 'has finite fibers' and we see that the convolution product (3) just becomes, for $f_{ij} \in \mathbb{C}(Z_{ij})$,

$$f_{12} * f_{23}(m_1, m_3) = \sum_{m_2 \in M_2} f_{12}(m_1, m_2) f_{23}(m_2, m_3).$$

In this spirit, we can consider the space X (resp. X') of n-step partial (resp. complete) flags on a d-dimensional vector space V over the field with q elements F_q. In the same way as in (13), we can parametrize $\mathrm{GL}(V)$-orbits in $X \times X$, $X' \times X'$ or $X \times X'$ respectively with the set M_d of $n \times n$ matrices with non-negative integer entries with sum of entries equal to d; with $d \times d$ permutation matrices, that we identify with the symmetric group S_d; the set Π of $n \times d$ matrices with exactly one entry 1 in each column and zeros elsewhere.

Now, we have the vector spaces $\mathbb{C}(\mathrm{M}_d)$, $\mathbb{C}(S_d)$ and $\mathbb{C}(\Pi)$, which we can look at as the subspaces of the $\mathbb{C}$-valued functions on $X \times X$, $X' \times X'$ and $X \times X'$ consisting on the ones which are constant along $\mathrm{GL}(V)$-orbits.

Definition 7.1. We define a convolution product on $\mathbb{C}(\mathrm{M}_d)$, for $F, G \in X$,

$$\varphi * \psi(F,G) = \sum_{L \in X} \varphi(F,L)\psi(L,G) \tag{14}$$

where it is clear that if $\varphi, \psi \in \mathbb{C}(X \times X)$ are constant on the orbits, so is $\varphi * \psi$.

With the exact same formula, if $\varphi, \psi \in \mathbb{C}(X' \times X')$ and the sum is taken over $L \in X'$ we have a product on $\mathbb{C}(S_d)$. Further, if $\varphi \in \mathbb{C}(\mathrm{M}_d)$ and $\psi \in \mathbb{C}(\Pi)$, we can see that (14) gives a left $\mathbb{C}(\mathrm{M}_d)$-action on $\mathbb{C}(\Pi)$ and analogously we can define a right $\mathbb{C}(S_d)$-action on $\mathbb{C}(\Pi)$.

Notice that, when we compute the products in $\mathbb{C}(\mathrm{M}_d)$ using (14) we will get coefficients in $\mathbb{Z}[q]$, so if we take q to be a formal parameter $\mathbf{q}$ we can see that those become the structure constants of a semisimple $\mathbb{C}(\mathbf{q})$-algebra $\mathbf{A}$ with M_d as a $\mathbb{C}(\mathbf{q})$-basis. In the same way we get a semisimple $\mathbb{C}(\mathbf{q})$-algebra H with S_d as a $\mathbb{C}(\mathbf{q})$-basis and a $(\mathbf{A}, H)$-bimodule structure on the $\mathbb{C}(\mathbf{q})$ vector space $\mathbf{T}$ spanned by Π.

Remark 7.2. The algebra H is the Iwahori-Hecke algebra corresponding to S_d and $\mathbf{A}$ is the algebra defined in [BLM, 1.2], which is a quotient of the quantized enveloping algebra $\mathbf{U}$.

This tells us that in the limit $\mathbf{q} = 1$ these algebras reduce to the group algebra of S_d and to the quotient of $\mathcal{U}(\mathfrak{sl}_n)$ that we constructed in section 5, hence giving us another construction of those. Now, let us consider the $\mathfrak{sl}_n$-action on $\mathbb{C}^n$ and the induced $\mathcal{U}(\mathfrak{sl}_n)$-action on $(\mathbb{C}^n)^{\otimes d}$. Let $I_d \subset \mathcal{U}(\mathfrak{sl}_n)$ be the annihilator of $(\mathbb{C}^n)^{\otimes d}$, which is a two sided ideal of finite codimension; we can improve the statement of Theorem 5.5 to $H(Z) \simeq \mathcal{U}(\mathfrak{sl}_n)/I_d$. We also have the right S_d-action on $(\mathbb{C}^n)^{\otimes d}$ by permutations of the coordinates. The following is a classical result of H. Weyl which is at the origin of Schur-Weyl duality.

Proposition 7.3. *The images of $\mathcal{U}(\mathfrak{sl}_n)$ and $\mathbb{C}[S_d]$ in $\mathrm{End}((\mathbb{C}^n)^{\otimes d})$ commute and we have an algebra isomorphism*

$$\mathrm{End}_{S_d}((\mathbb{C}^n)^{\otimes d}) = \mathcal{U}(\mathfrak{sl}_n)/I_d.$$

We can state a $\mathbf{q}$-deformation analogue of that.

Proposition 7.4. *The quotient map $\mathbf{U} \to \mathbf{A}$ defines a $\mathbf{U}$-action on $\mathbf{T}$ and there is a Schur-Weyl duality between the actions of $\mathbf{U}$ and H, that is their respective images in $\mathrm{End}(\mathbf{T})$ are the full centralizers of each other.*

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