Holomorphic PDE and Random Projections

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Abstract.

Consider two random subspaces $V, W \subseteq \mathbb{C}^d$. Then the dimension of $V \cap W$ is almost surely equal to $\max\{\dim V + \dim W - d, 0\}$ (i.e. they are in general position). Depending on the definition of ‘almost sure’, this easy theorem dates back 150 years or more. A more modern form could be stated thus: let $P, Q$ be random projection matrices on $\mathbb{C}^d$, and let $U_t$ be an independent Brownian motion on the unitary group $U(d, \mathbb{C})$. Conjugating $P$ by $U_t$ performs a random rotation of the subspace $P(\mathbb{C}^d)$; the general position statement can be written as $\text{Tr}[U_t P U_t^* \wedge Q] = \text{Tr}P + \text{Tr}Q - d$ a.s. for all $t > 0$ (here $P \wedge Q$ is the projection onto $P(\mathbb{C}^d) \cap Q(\mathbb{C}^d)$).

What happens when $d \to \infty$? It is still possible to make sense of the unitary Brownian motion and the trace for (some) projections on infinite-dimensional Hilbert spaces, but the easy techniques for proving the general position theorem are unavailable. Instead, one can analyze the spectral measure $\mu_t$ of the operator $U_t P U_t^* Q$ for smoothness. The Cauchy transform $G(t, z)$ of $\mu_t$ satisfies a holomorphic PDE in the upper-half-plane, locally similar to the complex inviscid Burger’s equation:

$$\frac{\partial}{\partial t} G(t, z) = \frac{\partial}{\partial z} \left[ z(z-1)G(t, z)^2 + (az+b)G(t, z) \right].$$

I will discuss the analysis of the characteristics of this PDE, and prove not only the general position claim in infinite-dimensions but a very strong smoothing theorem that settles (a special case of) an important conjecture in free probability theory.

This is joint work with Benoit Collins.