Quasi-solution approach towards nonlinear problems

Saleh Tanveer (The Ohio State University)

Abstract. Strongly nonlinear problems, written abstractly in the form $N[u] = 0$, are typically difficult to analyze unless they possess special properties. However, if we are able to find a quasi-solution $u_0$ in the sense that the residual $N[u_0] := R$ is small, then it is possible to analyze a strongly nonlinear problem with weakly nonlinear analysis in the following manner: We decompose $u = u_0 + E$; then $E$ satisfies $LE = -N_1[E] - R$, where $L$ is the Frechet derivative of the operator $N$ and $N_1[E] := N[u_0 + E] - N[u_0] - LE$ contains all the nonlinearity. If $L$ has a suitable inversion property and the nonlinearity $N_1$ is sufficiently regular in $E$, then weakly nonlinear analysis of the error $E$ through contraction mapping theorem gives rise to control of the error $E$. What is described above is quite routine. The only new element is to determine a quasi-solution $u_0$, which is typically found through a combination of classic orthogonal polynomial representation and exponential asymptotics.

This method has been used in a number of nonlinear ODEs arising from reduction of PDEs. We also show how it can be extended to integro-differential equations that arise in study of deep water waves of permanent form. The method is quite general and can in principle be applied to nonlinear PDEs as well.

Much of this is joint work with O. Costin and other collaborators.