$L^p$-resolvent estimates and density of eigenvalues for compact manifolds

Peng Shao
Department of Mathematics, Johns Hopkins University

Abstract. In 1987, Kenig, Ruiz and Sogge proved that the resolvent $(\Delta_{\mathbb{R}^n} + \zeta)^{-1}$ is uniformly bounded from $L^{\frac{2n}{n+2}}(\mathbb{R}^n)$ to $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ when $\zeta$ is outside a small disk centered at the origin in the complex plane. In 2001, Shen proved that for torus $\mathbb{T}^n$ the $L^{\frac{2n}{n+2}}(\mathbb{T}^n) \rightarrow L^{\frac{2n}{n-2}}(\mathbb{T}^n)$ uniform estimates also hold when $\zeta$ is outside a disk and a parabola. Recently Dos Santos Ferreira, Kenig and Salo extended this result to general compact Riemannian manifolds $(M, g)$ with $\zeta$ being in the same region as in Shen’s result. They also asked an interesting question about the best possible region $\mathcal{R}_g \subset \mathbb{C}, \zeta \in \mathcal{R}_g$ for which the uniform resolvent bounds $L^{\frac{2n}{n+2}}(M) \rightarrow L^{\frac{2n}{n-2}}(M)$ may hold.

During this talk I will present a recent joint work with J. Bourgain, C. D. Sogge and X. Yao which addresses this question. By establishing sharp bounds based on the distance of $\zeta$ to the spectrum, we showed that the parabolic boundary for the region is in fact optimal for round spheres $S^n$. This optimum is due to the high concentration phenomenon of the eigenvalues of the spheres, which indicates a potential relation between the density of eigenvalues and $L^p$ resolvent estimates for a general compact manifold may exist. This was consequently confirmed in our paper by revealing the equivalence between shrinking spectral projection estimates and improved resolvent estimates on compact manifolds. Based on this relation, we showed that the region $\mathcal{R}_g \subset \mathbb{C}$ can be improved logarithmically for manifolds with non-positive curvature, and by powers for the flat torus. The best improvement on the torus so far we can obtain is based on Bourgain’s recently-developed techniques (with Guth) using multilinear estimates of Bennett, Carbery and Tao.