Application of discrete geometry to the construction of Laurent-rational zeros*

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Abstract

We consider zeros of polynomials whose coefficients lie in the field $\mathbb{C}((t))$ of formal Laurent series with complex coefficients. The algebraic closure of $\mathbb{C}((t))$ is the field $\mathbb{K}$ of "Puiseux series," which allow fractional exponents. There is a well-known algorithm, described in [9], for constructing roots in $\mathbb{K}$ to a one-variable polynomial over $\mathbb{C}((t))$. Several papers give generalizations of this algorithm; for instance, [5] gives an algorithm for constructing zeros to systems of multivariable polynomials over $\mathbb{C}((t))$. We generalize the one-variable algorithm to multivariable polynomials with the specific goal of bounding the degree of the field extension over $\mathbb{C}((t))$ in which the specified zeros lie. We adapt recent techniques from tropical geometry which involve discrete and piecewise-linear geometry.

1 Introduction

The *field of formal Laurent series with complex coefficients*, denoted $\mathbb{C}((t))$, is the set of all formal series of the form $\sum_{i=b}^{\infty} c_i t^i$, where $b \in \mathbb{Z}$ and every $c_i \in \mathbb{C}$. The field of formal Laurent series is not algebraically closed; for instance, it contains no N^{th} root of t for $N \geq 2$. Furthermore, there exist multivariable polynomials such as $1 + tx^3 + t^2y^3$ with no zeros in $\mathbb{C}((t))^2$; see Example 4.7. For any $f(x_1, \ldots, x_n) = f(\mathbf{x}) \in \mathbb{C}((t))[\mathbf{x}]$ not a monomial, there exist finite extensions $\Bbbk$ over $\mathbb{C}((t))$ such that $(\Bbbk^*)^n$ contains a zero of f. The purpose of this paper is to establish an effectively computable bound on the minimum natural number d such that some extension $\Bbbk$ of degree d has this property. We show that this bound is sharper than certain naïve bounds, including naïve uses of Hensel's Lemma, and we give an infinite family of cases in which it is strictly sharper.

To state the bound requires certain notions from combinatorial geometry.

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Definition 1.1. Let $f(x_1, \ldots, x_n) \in \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, for some field $\Bbbk$. Then there exists a unique finite set $\mathcal{A} \subset \mathbb{Z}^n$ that indexes nonzero coefficients $a_{\mathbf{i}} \in \Bbbk^*$ such that

$$f(\mathbf{x}) = \sum_{\mathbf{i} \in \mathcal{A}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}.$$

$\mathcal{A}$ is called the *monomial support* for f, and may be denoted $\operatorname{msupp}(f)$. The *Newton polytope* of f, denoted $\operatorname{New}(f)$, is the convex hull of the monomial support in $\mathbb{R}^n$. If ord is a non-Archimedean valuation on $\Bbbk$, then the *valuated monomial support* $\operatorname{msupp}_{\operatorname{ord}}(f)$ of f is $\{(\mathbf{i}, \operatorname{ord}(a_{\mathbf{i}}))\}_{\mathbf{i} \in \mathcal{A}} \subset \mathbb{Z}^n \times \mathbb{R}$ and the *valuated Newton polytope* is its convex hull.

Definition 1.2. Let $\mathcal{D}$ be a compact non-empty subset of $\mathbb{R}^n$. The *dimension* of $\mathcal{D}$ is the dimension of its affine span. Thus, any set containing more than one point has nonzero dimension. We say $\mathcal{D}$ is *trivial* if its dimension is zero. The *face* of $\mathcal{D}$ with respect to $\mathbf{n} \in \mathbb{R}^n$ is

$$\operatorname{face}(\mathbf{n}, \mathcal{D}) = \{\mathbf{u} \in \mathcal{D} \colon \text{ for all } \mathbf{v} \in \mathcal{D},\ \mathbf{n} \cdot \mathbf{u} \leq \mathbf{n} \cdot \mathbf{v}\}.$$

A subset of $\mathcal{D}$ is said to be a face of $\mathcal{D}$ if it is the face with respect to some vector.

Definition 1.3. Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite one-dimensional set with endpoints $\mathbf{a}, \mathbf{b}$. Let $\mathbf{v} \in \mathbb{Z}^n$ be the unique maximal-length vector such that $\mathcal{A} \subset \mathbf{a} + \mathbb{Z}_{\geq 0}\mathbf{v}$. The *lattice sublength* of $\mathcal{A}$ is the natural number q such that $\mathbf{b} = \mathbf{a} + q\mathbf{v}$.

Definition 1.4. The *pseudo-multiplicity* $\operatorname{p}\underline{\operatorname{mult}}(\mathcal{A})$ of a non-trivial finite set $\mathcal{A} \subset \mathbb{Z}^n$ is the minimum of the lattice sublengths of its one-dimensional faces.

Our main result is the following.

Theorem 1.5. Let $f(\mathbf{x}) \in \mathbb{C}((t))[\mathbf{x}]$ be a polynomial with nontrivial valuated monomial support $\mathcal{A}$. Let $\boldsymbol{\gamma} \in \left(\frac{1}{N_1}\mathbb{Z}\right)^n$ and suppose $\operatorname{face}\big((\boldsymbol{\gamma}, 1), \mathcal{A}\big)$ is a nontrivial set with pseudo-multiplicity M. Then f has a zero in $(\Bbbk^*)^n$, for some extension $\Bbbk$ of degree at most $N_1 M$.

The proof of the main theorem, deferred to sections 8 and 9, generalizes a well-known algorithm described on pp. 97-102 of [9] for constucting roots to one-variable Laurent-series polyonomials. Several papers give generalizations of this algorithm; for instance, [5] gives an algorithm for constructing zeros to systems of multivariable polynomials over $\mathbb{C}((t))$. These papers commonly employ recent techniques such as tropical geometry that translate problems of algebraic geometry into problems of combinatorial geometry. Such techniques are described in [3] and in [8]; the slightly unusual introduction of "face" given above comes from Chapter 2 of [8], with a few alterations. In this paper, to be as conservative as possible with the degree of the field extension, we adapt techniques that are typically used with the Newton polytope to apply to the monomial support instead.

2 Naïve Bounds

2.1 Hensel's Lemma

Definition 2.1. Let $\mathbb{K}$ denote the *field of formal Puiseux series with complex coefficients*, defined as $\bigcup_{N\in\mathbb{N}} \mathbb{C}((t^{1/N}))$.

The Puiseux series field is algebraically closed (see [9]). Considering the possible subfields of $\mathbb{K}$ and their behavior under automorphisms $t^{1/N} \mapsto e^{2k\pi i/N} t^{1/N}$ gives a concrete characterization for finite extensions of $\mathbb{C}((t))$.

Proposition 2.2. Every finite extension of $\mathbb{C}((t))$ is of the form $\mathbb{C}((t^{1/N}))$, where N is the degree of the extension.

Remark 2.3. Every nonzero Puiseux series $s \in \mathbb{K}^*$ may be written uniquely as a possibly finite series

$$s = c_1 t^{\gamma_1} + c_2 t^{\gamma_2} + c_3 t^{\gamma_3} + \cdots,$$

where $c_i \in \mathbb{C}^*$, $\gamma_i \in \frac{1}{N}\mathbb{Z}$ for some N that does not depend on i, and the γ_i are strictly increasing.

Definition 2.4. Given a nonzero Puiseux series $s \in \mathbb{K}^*$ written as in Remark 2.3, its *order*, denoted $\operatorname{ord}(s)$, is the leading exponent γ_1. The order map may be extended componentwise to n-tuples, giving $\operatorname{ord}\colon (\mathbb{K}^*)^n \to \mathbb{R}^n$.

The order map gives a non-Archimedean valuation on algebraic extensions of $\mathbb{C}((t))$. Note that

$$\operatorname{ord}\big(\mathbb{C}((t^{1/N}))^*\big) = \tfrac{1}{N}\mathbb{Z}$$
$$\operatorname{ord}(\mathbb{K}^*) = \mathbb{Q}.$$

When constructing zeros in complete rings, an obvious tool to use is Hensel's Lemma. One version, based on Theorem 7.3 of [2], is the following.

Theorem 2.5. If $f \in \mathbb{C}[[t]][\mathbf{x}]$, and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}[[t]]^n$ satisfies

$$\operatorname{ord}(f(\mathbf{a})) > 2 \operatorname{ord}\left(\frac{\partial f}{\partial x_k}(\mathbf{a})\right)$$

for some k, then there exists $b \in \mathbb{C}[[t]]$ such that

$$\operatorname{ord}(a_k - b) > \operatorname{ord}\left(\frac{\partial f}{\partial x_k}(\mathbf{a})\right)$$

and $f(a_1, \ldots, a_{k-1}, b, a_{k+1}, \ldots, a_n) = 0$.

There are simple ways to extend this theorem to cases it does not directly cover. For instance, for $\mathbf{a} \in \mathbb{C}((t))$, we may replace f by an appropriate $f(\mathbf{x}t^{\gamma})$. To detect zeros in extension fields, we may replace t in the statement of the theorem by $T = t^{1/d}$. Our purpose is to establish a family of polynomials of which no "reasonable" use of 2.5 can detect any zero.

By a "reasonable" use of the theorem, we mean an argument that constructs an n-tuple of series $\mathbf{a} \in \mathbb{C}((T))^n$ and then, from certain hypotheses on $\mathbf{a}$ and on the specified polynomial $f \in \mathbb{C}((T))[\mathbf{x}]$, concludes that f has a zero in $\mathbb{C}((T))^n$. Moreover, we assume that the hypotheses depend on only finitely many terms of f and $\mathbf{a}$ in the following sense: for sufficiently large m, if $\text{ord}(f - g) > m$ and $\text{ord}(b_i - a_i) > m$, then the hypotheses hold for $\mathbf{b}$ and g. As a consequence, if a "reasonable" use of Hensel's lemma shows that a given polynomial f has a zero, then for some m, the same argument shows that every polynomial g such that $\text{ord}(f - g) > m$ also has a zero. It is not difficult to verify that a direct application of the statement of 2.5 is "reasonable" by this definition.

Consider a polynomial of the form $f(\mathbf{x}) = (p(\mathbf{x}))^r$, where $r \geq 2$ and $p(\mathbf{x}) \in \mathbb{C}((T))[\mathbf{x}]$ is a non-constant polynomial. We claim that no "reasonable" use of Hensel's Lemma can detect a zero of f in $\mathbb{C}((T))^n$. If we had a valid "reasonable" use, then there would be some m such that, for every polynomial $g \in \mathbb{C}((T))[\mathbf{x}]$ such that $\text{ord}(f - g) > m$, g has a root in $\mathbb{C}((T))^n$. Consider the polynomial $g(\mathbf{x}) = p(\mathbf{x})^r - T^d$, where $d > m$ and d is not a multiple of r. Then g has no zero in $\mathbb{C}((T))^n$, a contradiction. Note that we may generalize this argument to apply to all f of inferior multiplicity greater than one with respect to the ring $\mathbb{C}((T))[\mathbf{x}]$, in the sense of Definition 7.2.

Now, for the polynomials f just described, when these are considered as polynomials over $\mathbb{C}((T))$, no technique presented in this paper can typically detect zeros in $\mathbb{C}((T))^n$. What we are able to do is to specify low-degree extensions over $\mathbb{C}((T))$ that are guaranteed to contain zeros. The only "reasonable" way for Hensel's Lemma to deal with field extensions is to regard f as a polynomial over an extension field, in which case the same argument shows that Hensel's Lemma can detect no zeros in this extension field.

Metamathematical arguments of this sort must be treated with caution, since the argument is only valid to the extent that the definition of "reasonable" use of Hensel's Lemma actually applies to arguments that use the lemma. Nevertheless, we believe that this line of reasoning gives some justification for our claim that the results of this paper represent an improvement over Hensel's Lemma.

2.2 Lattice Width

Let $\mathbf{i}$ be an n-tuple $(i_1, \ldots, i_n)$ of integers. The symbol $\mathbf{x}^\mathbf{i}$ will be used for the product monomial $x_1^{i_1} \cdots x_n^{i_n}$. In the same vein, for vectors $\mathbf{a} \in (\mathbb{k}^*)^n$, the symbol $\mathbf{a}^\mathbf{i}$ will denote the product $a_1^{i_1} \cdots a_n^{i_n} \in \mathbb{k}^*$.

Definition 2.6. Let A be an $n \times m$ integer matrix. The *monomial morphism* associated to A over a field $\mathbb{k}$ is the ring homomorphism

$$\Phi_A \colon \mathbb{k}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] \to \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \qquad \text{determined by}$$
$$\Phi_A(\mathbf{x}^\mathbf{i}) = \mathbf{x}^{A\mathbf{i}}.$$

A monomial morphism Φ_A is a ring isomorphism if and only if its matrix A is unimodular, i.e., has determinant ± 1. In this case we call Φ_A a *monomial isomorphism*.

For each monomial morphism Φ_A there is an associated map $\phi_A\colon (\Bbbk^*)^n \to (\Bbbk^*)^m$ such that

$$\Phi_A(f)(\mathbf{a}) = f\big(\phi_A(\mathbf{a})\big). \tag{1}$$

This map is given by $\mathbf{a} \mapsto (\mathbf{a}^{A\mathbf{e}_1}, \ldots, \mathbf{a}^{A\mathbf{e}_m})$, where $\mathbf{e}_i$ denotes the i^{th} standard basis vector.

Definition 2.7. The *lattice width* of a nontrivial finite set $\mathcal{A} \subset \mathbb{Z}^n$ is the minimum value of $\max\big(\tau(\mathcal{A})\big) - \min\big(\tau(\mathcal{A})\big)$ over all $\mathbb{Z}$-module homomorphisms $\tau\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $\tau(\mathcal{A})$ is nontrivial.

Clearly, in computing lattice width, we may without loss of generality restrict to surjective homomorphisms τ.

Proposition 2.8. Suppose $f \in \mathbb{C}((t))[\mathbf{x}]$ is an n-variable polynomial with nontrivial monomial support $\mathcal{A}$. Then $f(\mathbf{x})$ has a zero in $(\Bbbk^*)^n$ for some field extension $\Bbbk$ over $\mathbb{C}((t))$ of degree less than or equal to the lattice width of $\mathcal{A}$.

Proof. Let d be the lattice width of $\mathcal{A}$, and suppose $\tau\colon \mathbb{Z}^n \to \mathbb{Z}$ is a surjective $\mathbb{Z}$-module homomorphism such that $\max(\tau(\mathcal{A})) - \min(\tau(\mathcal{A})) = d$. Let $\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\ker(\tau)$ and let $\mathbf{v}_1 \in \tau^{-1}(\{1\})$. Then $B := \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ generate $\mathbb{Z}^n$. Since B also spans $\mathbb{Q}^n$ as a $\mathbb{Q}$-vector space, there can be no linear dependence among the elements of B. Hence, B is a basis for $\mathbb{Z}^n$. Let $A\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be the isomorphism defined by setting $A(\mathbf{v}_i) = \mathbf{e}_i$ and extending by linearity. Then $\tau = \pi_1 A$, where $\pi_1\colon \mathbb{Z}^n \to \mathbb{Z}$ is projection on the first coordinate. Let $\mathbf{i} = -\min(\tau(\mathcal{A}))\mathbf{v}_1$, so that $\min\big(\tau(\mathbf{i} + \mathcal{A})\big) = 0$. Then

$$\begin{aligned}
\deg_{x_1}\big(\Phi_A(\mathbf{x}^{\mathbf{i}} f)\big) &= \max\big(\pi_1 A(\mathbf{i} + \mathcal{A})\big) \\
&= \max\big(\tau(\mathbf{i} + \mathcal{A})\big) \\
&= \max(\tau(\mathcal{A})) - \min(\tau(\mathcal{A})) \\
&= d.
\end{aligned}$$

We may choose $\bar{x}_2, \ldots, \bar{x}_n \in \mathbb{C}((t))^*$ such that the one-variable polynomial $g(x_1) := \Phi_A(\mathbf{x}^{\mathbf{i}} f)(x_1, \bar{x}_2, \ldots, \bar{x}_n) \in \mathbb{C}((t))[x_1]$ is not a monomial. Then for some extension $\Bbbk$ over $\mathbb{C}((t))$ of degree at most d, g has a root $\bar{x}_1 \in \Bbbk^*$. Therefore, $\bar{\mathbf{x}} := (\bar{x}_1, \ldots, \bar{x}_n) \in \Bbbk^*$ is a zero of $\Phi_A(\mathbf{x}^{\mathbf{i}} f)$. By (1), $\phi_A(\bar{\mathbf{x}})$ is the desired zero of f. ☐

We show in section 6 that the bound of Theorem 1.5 is sharper than the lattice width, and is strictly sharper for an infinite family of polynomials.

3 Geometric Preliminaries

3.1 Convex and discrete geometry

Let $\mathcal{D}$ be a nonempty compact set in $\mathbb{R}^n$. For our purposes, $\mathcal{D}$ will be either a finite set of points or the convex hull of a finite set of points.

Definition 3.1. If $\mathcal{F}$ is a face of $\mathcal{D}$ and $\mathbf{n}$ satisfies $\mathcal{F} \subset \text{face}(\mathbf{n}, \mathcal{D})$, we say that $\mathbf{n}$ is an *inward normal* at $\mathcal{F}$.

Remark 3.2. For any two points $\mathbf{a}, \mathbf{b} \in \text{face}(\mathbf{n}, \mathcal{D})$, $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{n} = 0$; i.e., the inward normal is orthogonal to $(\mathbf{a} - \mathbf{b})$.

If $\mathcal{P}$ is the convex hull of $\mathcal{D}$, we write $\mathcal{P} = \text{conv}(\mathcal{D})$.

Definition 3.3. We say that $\mathcal{P}$ is a *convex polytope* if it is the convex hull of a finite set of points. A face of a convex polytope is called a *vertex* if it is trival and an *edge* if it is one-dimensional. The *relative interior* of a face of a polytope is the set of all points on the face that are not on any lower-dimensional face.

If $\mathcal{F}$ is a face of a polytope $\mathcal{P}$, $\mathbf{a}$ is a point on its relative interior, and $\mathbf{a} \in \text{face}(\mathbf{n}, \mathcal{P})$ for a given $\mathbf{n} \in \mathbb{R}^n$, then $\mathcal{F} \subset \text{face}(\mathbf{n}, \mathcal{P})$.

We will use $\mathbf{i} = (i_1, \dots, i_n)$ to denote points in $\mathbb{R}^n$ and we will use $(\mathbf{i}, w) = (i_1, \dots, i_n, w)$ to denote points in $\mathbb{R}^{n+1}$. We will make use of the projections

$$\begin{aligned} \pi \colon \mathbb{R}^{n+1} &\to \mathbb{R}^n \\ (i_1, \dots, i_n, w) &\mapsto (i_1, \dots, i_n) \end{aligned} \qquad\qquad \begin{aligned} \pi_k \colon \mathbb{R}^{n+1} &\to \mathbb{R} \\ (i_1, \dots, i_n, w) &\mapsto i_k. \end{aligned}$$

The operations of taking a projection and taking the face with respect to a particular vector each commute with taking the convex hull.

Note that, for a Laurent-series polynomial f, the projection π induces a bijection from the valuated monomial support $\text{msupp}_{\text{ord}}(f)$ to the monomial support $\text{msupp}(f)$.

Definition 3.4. The *lattice length* of an integer lattice vector $\mathbf{a} \in \mathbb{Z}^n$, which will be denoted as $\text{ll}(\mathbf{a})$, is the greatest integer d such that $\frac{1}{d}\mathbf{a} \in \mathbb{Z}^n$, or zero if $\mathbf{a} = \mathbf{0}$. The lattice length of a line segment with endpoints $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^n$ is the lattice length of the integer lattice vector $\mathbf{j} - \mathbf{i}$. The lattice length of a finite one-dimensional set $\mathcal{S} \subset \mathbb{Z}^n$ is the lattice length of the line segment $\text{conv}(\mathcal{S})$.

Remark 3.5. The lattice length of a line segment with endpoints in $\mathbb{Z}^n$ is one less than the number of integer lattice points on the line segment. The lattice length of an integer lattice vector $\mathbf{a} = (a_1, \dots, a_n)$ is equal to the greatest common divisor of the integers $a_1, \dots, a_n$. The lattice length of a finite one-dimensional set in $\mathbb{Z}^n$ is less than or equal to its lattice sublength.

3.2 Multiplicities

Definition 3.6. A Laurent polynomial $f(x_1, \dots, x_n) \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is said to be *normalized* if the following conditions are satisfied:

1. The polynomial contains no negative powers, that is, $f(x_1, \ldots, x_n) \in \Bbbk[x_1, \ldots, x_n]$.
2. In the polynomial ring $\Bbbk[x_1, \ldots, x_n]$, for each $k \in \{1, \ldots, n\}$, we have that $x_k \nmid f(x_1, \ldots, x_n)$.

For each nonzero multivariate Laurent polynomial $f(\mathbf{x}) \in \Bbbk[\mathbf{x}^{\pm 1}]$, there is a unique $\mathbf{i} \in \mathbb{Z}^n$ such that $\mathbf{x}^{\mathbf{i}} f(\mathbf{x})$ is normalized. We say that $\mathbf{x}^{\mathbf{i}} f(\mathbf{x})$ is the *normalization* of f.

Definition 3.7. Let $\mathcal{A} \subset \mathbb{Z}^n$ be a monomial support. The *degree* of $\mathcal{A}$ is the total degree of the normalization of f, for any Laurent polynomial $f(x_1, \ldots, x_n)$ with monomial support $\mathcal{A}$.

If $\mathcal{A}$ is a monomial support and $\mathcal{S} \subset \mathcal{A}$, then $\deg(\mathcal{S}) \leq \deg(\mathcal{A})$. A monomial support is nontrivial if and only if its degree is at least one.

Definition 3.8. Let $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$, and let $\mathbf{a} \in \mathbb{C}^n$. Write

$$f(\mathbf{x}) = \sum_{k=0}^{\deg(f)} f_k(\mathbf{x} - \mathbf{a}),$$

where f_k is homogeneous of degree k. The *multiplicity* of $\mathbf{a}$, denoted $\operatorname{mult}_f(\mathbf{a})$, is the least k such that $f_k \neq 0$.

The multiplicity of a point cannot exceed the degree of f. By the properties of graded rings, multiplicities are additive, in the sense that

$$\operatorname{mult}_{f \cdot g}(\mathbf{x}) = \operatorname{mult}_f(\mathbf{x}) + \operatorname{mult}_g(\mathbf{x}).$$

Definition 3.9. Let $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with at least one zero in $(\mathbb{C}^*)^n$. Then the *inferior multiplicity* of f, denoted $\underline{\operatorname{mult}}(f)$, is the minimum of the multiplicities of all zeroes in $(\mathbb{C}^*)^n$; i.e.,

$$\underline{\operatorname{mult}}(f) = \min\{\operatorname{mult}_f(\mathbf{x})\colon \mathbf{x} \in (\mathbb{C}^*)^n \text{ and } f(\mathbf{x}) = 0\}.$$

Suppose two polynomials in $\mathbb{C}[x_1, \ldots, x_n]$, each with at least two terms, differ by a factor of $x_1^{i_1} \cdots x_n^{i_n}$, for some $(i_1, \ldots, i_n) \in \mathbb{Z}^n$. By additivity of multiplicities, the two polynomials have the same inferior multiplicity. In particular, the inferior multiplicity of a polynomial is the same as the inferior multiplicity of its normalization. Thus, the following definition of the inferior multiplicity of a Laurent polynomial is consistent with Definition 3.9 in the case of Laurent polynomials with no negative exponents.

Definition 3.10. Let $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ be a complex Laurent polynomial with at least two terms. The *inferior multiplicity* of f, denoted $\underline{\operatorname{mult}}(f)$, is the inferior multiplicity of the normalization of f as defined in Definition 3.9.

Definition 3.11. Given a nontrivial finite set $\mathcal{A} \subset \mathbb{Z}^n$, its *inferior multiplicity* is the maximum inferior multiplicity of all complex Laurent polynomials with monomial support $\mathcal{A}$; i.e.,

$$\underline{\text{mult}}(\mathcal{A}) = \max\{\underline{\text{mult}}(f)\colon\ \text{msupp}(f) = \mathcal{A} \text{ and } f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\}.$$

The inferior multiplicity of a monomial support might be thought of as the "best point on the worst hypersurface."

Remark 3.12. The inferior multiplicity of any nontrivial monomial support $\mathcal{A}$ is less than or equal to its degree; i.e., $\underline{\text{mult}}(\mathcal{A}) \leq \deg(\mathcal{A})$.

The proofs of the following two theorems are deferred to section 7.

Theorem 3.13. Let $\mathcal{A}$ be a nontrivial monomial support. Let $\mathfrak{F}$ be the set of all faces that are nontrivial (and hence for which inferior multiplicity is defined). Then

$$\underline{\text{mult}}(\mathcal{A}) \leq \min_{\mathcal{F} \in \mathfrak{F}} \underline{\text{mult}}(\mathcal{F}).$$

Theorem 3.14. (Lazy Lemma) Let $\mathcal{A} \subset \mathbb{Z}^n$ be a nontrivial monomial support, and let $\mu\colon \mathbb{Q}^n \to \mathbb{Q}^n$ be a bijective affine map such that $\mu(\mathcal{A}) \subset \mathbb{Z}^n$. Then

$$\underline{\text{mult}}(\mu(\mathcal{A})) = \underline{\text{mult}}(\mathcal{A}).$$

Corollary 3.15. (a) The inferior multiplicity of a one-dimensional monomial support is less than or equal to its lattice sublength.

(b) The inferior multiplicity of any nontrivial finite set $\mathcal{D} \subset \mathbb{Z}^n$ is less than or equal to its pseudo-multiplicity.

Proof. (a) Let $\mathcal{A}$ be a one-dimensional monomial support with endpoints $\mathbf{a}$, $\mathbf{b}$. Suppose $\mathbf{v} \in \mathbb{Z}^n$ satisfies $\mathcal{A} \subset \mathbf{a} + \mathbb{Z}_{\geq 0}\mathbf{v}$, and in particular $\mathbf{b} = \mathbf{a} + q\mathbf{v}$. Let $\mu\colon \mathbb{Q}^n \to \mathbb{Q}^n$ be a bijective affine map that takes $\mathbf{a}$ to $\mathbf{0}$ and $\mathbf{v}$ to $(1, 0, \ldots, 0)$. Thus, $\mu(\mathcal{A}) \subset \mathbb{Z}^n$. Since $\mu(\mathbf{b}) = (q, 0, \ldots, 0)$, $\deg\big(\mu(\mathcal{A})\big) = q$. By Remark 3.12 and the Lazy Lemma, $\underline{\text{mult}}(\mathcal{A}) \leq q$.

(b) This follows from (a), Theorem 3.13, and the definition of pseudo-multiplicity. ∎

4 The Tropical Hypersurface

Given an n-variable Puiseux-series polynomial $f(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$, let $\mathcal{P} \subset \mathbb{R}^{n+1}$ denote its valuated Newton polytope.

Definition 4.1. A face $\mathcal{F}$ of $\mathcal{P}$ is a *lower face* of $\mathcal{P}$ if $\mathcal{F}$ has an inward normal $\mathbf{n}$ with final coordinate $w_{\mathbf{n}}$ equal to 1.

The existence of an inward normal with final coordinate 1 is equivalent to the existence of an inward normal with final coordinate positive. The lower faces of $\mathcal{P}$ are the faces that are "visible from below."

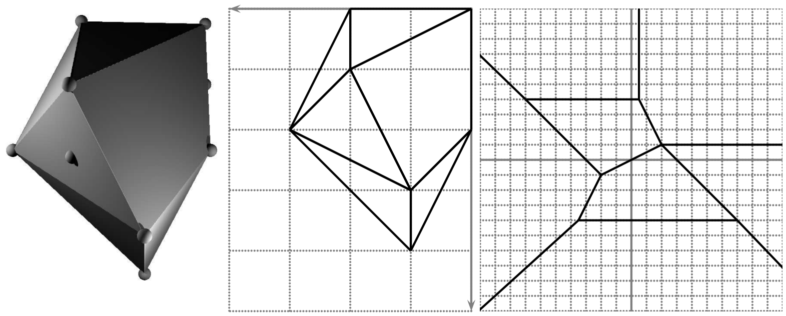

Figure 1: The normal and subdivision complexes

Definition 4.2. Given a lower face $\mathcal{F}$ of $\mathcal{P}$, the *subdivision cell* $\mathfrak{s}_{\mathcal{F}}$ corresponding to $\mathcal{F}$ is the projection of $\mathcal{F}$ down onto $\mathbb{R}^n$ upon elimination of the final coordinate: $\mathfrak{s}_{\mathcal{F}} = \pi(\mathcal{F})$.

The *subdivision complex* $\Delta_s(f)$ is the collection of the subdivision cells $\mathfrak{s}_{\mathcal{F}}$ corresponding to all the lower faces $\mathcal{F}$ of $\mathcal{P}$. The union of the subdivision cells is the Newton polytope $\text{New}(f) = \pi(\mathcal{P})$, much as the union of the faces of $\mathcal{P}$ is $\mathcal{P}$ itself[1].

The set of all inward normals to a face $\mathcal{F}$ of $\mathcal{P}$ is a closed unbounded polyhedron called the *normal cone* of $\mathcal{P}$ at $\mathcal{F}$. The collection of all normal cones of $\mathcal{P}$ form a polyhedral complex whose union is $\mathbb{R}^{n+1}$.

Definition 4.3. For a lower face $\mathcal{F}$ of $\mathcal{P}$, the corresponding *normal cell* $\mathfrak{c}_{\mathcal{F}}$ is the intersection of the normal cone at $\mathcal{F}$ with the hyperplane $w = 1$. By definition of a lower face, this intersection is not empty.

The w-coordinate of points on a normal cell, being identically 1, is omitted.

The *normal complex* $\Delta_N(f)$ is the collection of the normal cells of $\mathcal{F}$. The union of all normal cells is $\mathbb{R}^n$, much as the union of all normal cones is $\mathbb{R}^{n+1}$. The normal complex is the dual complex of [6].

Example 4.4. Consider a two-variable polynomial of the form

$$\begin{aligned} f(x,y) = {} & a_{0,0}t^5 + a_{0,1}t^5y + a_{2,0}t^4x^2 + a_{0,2}t^3y^2 + a_{2,1}x^2y \\ & + a_{1,3}xy^3 + a_{2,2}tx^2y^2 + a_{3,2}t^3x^3y^2 + a_{1,4}t^4xy^4, \end{aligned}$$

where the $a_{i,j}$ are all Puiseux series of order 0. Figure 1 (left) illustrates the valuated Newton polytope $\mathcal{P}$ as seen from below. A small sphere surrounds each point of the valuated monomial support. The 1-skeletons of the subdivision and normal complexes (center and right, respectively) are shown. For the two leftmost images, the i- and j-axes have been inverted to make the duality with the normal complex more apparent.

[1]When $\pi(\mathcal{P})$ is formed as a union of projections of faces, the set of lower faces is sufficient, as would be the set of upper faces.

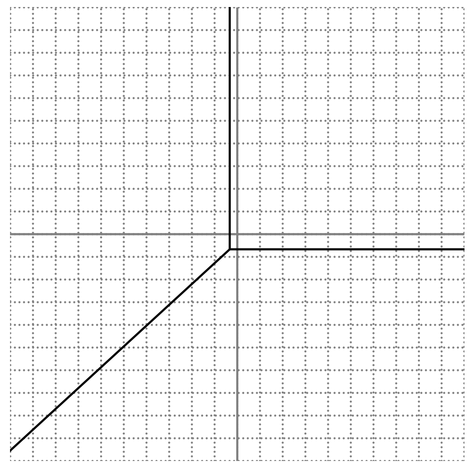

Figure 2: A tropical hypersurface with no lattice points

By construction, the normal and subdivision complexes are in bijective correspondence with the lower faces of $\mathcal{P}$, and hence with each other. This correspondence may be called a duality in that it is inclusion-reversing. The correspondence also has a number of useful geometric properties. If $\mathfrak{c}$ and $\mathfrak{s}$ are corresponding normal and subdivision cells, then $\dim(\mathfrak{s}) = n - \dim(\mathfrak{c})$. If $\mathbf{a}, \mathbf{b} \in \mathfrak{c}$ and $\mathbf{p}, \mathbf{q} \in \mathfrak{s}$, then $(\mathbf{a} - \mathbf{b})$ is normal to $(\mathbf{p} - \mathbf{q})$.

Definition 4.5. The *tropical hypersurface* of $f(\mathbf{x})$, denoted $\operatorname{trop}(f) \subset \mathbb{R}^n$, is the union of all normal cells of dimension less than n.

By the duality of the normal and subdivision complexes, a normal cell contributes to the tropical hypersurface if and only if the corresponding subdivision cell is nontrivial.

In the tropical geometry literature, it is more usual to define a tropical hypersurface as the nondifferentiability locus of a polynomial over the tropical semiring $(\mathbb{R} \cup \{\infty\}, \min, +)$; see [7].

The following theorem (Theorem 2.1.1 of [1]) shows the connection between the hypersurface of a polynomial (i.e., its zero locus) and its tropical hypersurface.

Theorem 4.6. (Kapranov) Let $f(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$ be an n-variable polynomial with Puiseux-series coefficients. The image of the hypersurface of f (excluding points with any coordinate zero) under the order map is the set of rational points on the tropical hypersurface:

$$\operatorname{ord}\bigl(\{\bar{\mathbf{x}} \in (\mathbb{K}^*)^n \colon f(\bar{\mathbf{x}}) = 0\}\bigr) = \operatorname{trop}(f) \cap \mathbb{Q}^n.$$

Example 4.7. In the Introduction, we claimed that the polynomial

$$f(x, y) = 1 + tx^3 + t^2y^3$$

has no Laurent-series zeros. Suppose $(\bar{x}, \bar{y}) \in \mathbb{C}((t))^2$ were a zero. Clearly, neither $\bar{x}$ nor $\bar{y}$ can be equal to 0. By Kapranov's Theorem, $\big(\mathrm{ord}(\bar{x}), \mathrm{ord}(\bar{y})\big)$ lies on the tropical hypersurface $\mathrm{trop}(f)$. Since $\bar{x}$, $\bar{y}$ are both Laurent series, their orders must be integers. This is a contradiction since $\mathrm{trop}(f)$ has no integer lattice points, as seen in Figure 2.

5 The Main Theorem

Definition 5.1. Let $\boldsymbol{\gamma} \in \mathrm{trop}(f)$, for some Puiseux-series polynomial $f \in \mathbb{K}[\mathbf{x}]$. We define the *inferior multiplicity* of $\boldsymbol{\gamma}$, denoted $\underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma})$, by

$$\underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma}) = \underline{\mathrm{mult}}\Big(\pi\Big(\mathrm{face}\big((\boldsymbol{\gamma}, 1), \mathrm{msupp}_{\mathrm{ord}}(f)\big)\Big)\Big).$$

Let us unwind this expression to make some sense of it.

The point $\boldsymbol{\gamma}$ lies in the relative interior of a unique normal cell $\mathfrak{c}$. By definition of the tropical hypersurface, $\dim(\mathfrak{c}) < n$. Let $\mathcal{F} = \mathrm{face}\big((\boldsymbol{\gamma}, 1), \mathcal{P}\big)$, where $\mathcal{P}$ is the valuated Newton polytope of $\mathcal{F}$. By duality $\mathcal{F}$ is nontrivial.

Let $\mathcal{F}' = \mathcal{F} \cap \mathrm{msupp}_{\mathrm{ord}}(f)$, so that

$$\mathcal{F}' = \mathrm{face}\big((\boldsymbol{\gamma}, 1), \mathrm{msupp}_{\mathrm{ord}}(f)\big) \subset \mathbb{Z}^n \times \mathbb{Q}.$$

When we project $\mathcal{F}'$ down onto $\mathbb{R}^n$, we obtain a finite subset $\pi(\mathcal{F}')$ of $\mathbb{Z}^n$ that defines a nontrivial monomial support. The inferior multiplicity of $\boldsymbol{\gamma}$ is the inferior multiplicity of $\pi(\mathcal{F}')$.

Similarly, tropical *pseudo-multiplicity*, denoted $\mathrm{p}\,\underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma})$, is defined as the pseudo-multiplicity of $\pi(\mathcal{F}')$. By Corollary 3.15, $\mathrm{p}\,\underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma}) \leq \underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma})$.

In the tropical geometry literature, it is more standard to use the term "multiplicity" for the following quantity, as described in Section 3 of [7].

Definition 5.2. Let $f(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$ be an n-variable polynomial with a normal cell $\mathfrak{c} \in \Delta_N(f)$ of dimension $n-1$. The corresponding subdivision cell $\mathfrak{s}$ is thus a line segment. The *multiplicity* of $\mathfrak{c}$ is the lattice length of $\mathfrak{s}$.

By definition of pseudo-multiplicity, the pseudo-multiplicity of a point $\boldsymbol{\gamma}$ is less than or equal to the multiplicity of any $(n-1)$-dimensional normal cell containing $\boldsymbol{\gamma}$.

The following theorem is provides a link between the techniques of discrete geometry and the paper's stated purpose.

Theorem 5.3. Let $f(\mathbf{x}) \in \mathbb{C}((t))[\mathbf{x}]$ be an n-variable Laurent-series polynomial. Suppose we are given a point $\boldsymbol{\gamma}$ on $\mathrm{trop}(f) \cap \mathbb{Q}^n$. Let $N_1 \in \mathbb{N}$ be the least natural number such that $\boldsymbol{\gamma} \in \left(\frac{1}{N_1}\mathbb{Z}\right)^n$, and let $M = \underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma})$. Then for some natural number $N \leq N_1 M$, f has a zero $\bar{\mathbf{x}} \in \left(\mathbb{C}((t^{1/N}))^*\right)^n$ such that $\mathrm{ord}(\bar{\mathbf{x}}) = \boldsymbol{\gamma}$.

The proof is constructive up to the construction of generic zeros to polynomials $\phi \in \mathbb{C}[\mathbf{x}]$; the details are deferred to sections 8 and 9. Since pseudo-multiplicity bounds inferior multiplicity, the Main Theorem follows as a corollary.

Theorem 5.4. (Main Theorem) Let $f(\mathbf{x})$, $\boldsymbol{\gamma}$, and N_1 be as in Theorem 5.3, and let $\widetilde{M} = \mathrm{p}\,\underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma})$. Then for some natural number $N \leq N_1\widetilde{M}$, f has a zero $\bar{\mathbf{x}} \in \left(\mathbb{C}((t^{1/N}))^*\right)^n$ such that $\mathrm{ord}(\bar{\mathbf{x}}) = \boldsymbol{\gamma}$.

Although the Main Theorem is slightly weaker than Theorem 5.3, it has the advantage that pseudo-multiplicity is effectively computable. It is an exercise in definition-chasing to show that, except for specifying the order of the zero, the Main Theorem is equivalent to Theorem 1.5.

6 An absolute bound on the degree of the field extension

The Main Theorem bounds the degree of the field extension by the product $N_1\widetilde{M}$, which depends on the choice of the tropical point $\boldsymbol{\gamma}$. In this section, we will show that, for a suitable choice of $\boldsymbol{\gamma}$, the bound $N_1\widetilde{M}$ is sharper than the lattice-width bound discussed earlier, and we give an infinite family of polynomials for which it is strictly sharper.

6.1 Beating the Naïve Bound

Proposition 6.1. Suppose $f \in \mathbb{C}((t))[x_1, \ldots, x_n]$ is a Laurent-series polynomial whose monomial support has lattice width $d > 0$. Then there exists a tropical point $\boldsymbol{\gamma} \in \mathrm{trop}(f)$ such that $\boldsymbol{\gamma} \in \left(\frac{1}{N_1}\mathbb{Z}\right)^n$, for some natural number N_1 satisfying

$$N_1 \leq \frac{d}{\mathrm{p}\,\underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma})}.$$

When the Main Theorem is applied to the point $\boldsymbol{\gamma} \in \mathrm{trop}(f)$ whose existence is guaranteed by Proposition 6.1, the resulting zero lies in $\left(\mathbb{C}((t^{1/N}))^*\right)^n$, for some natural number $N \leq d$. This is precisely the naïve bound on the degree of the field extension given by Proposition 2.8.

We now prove Proposition 6.1.

Lemma 6.2. Let $\mathcal{S}'$ be a one-dimensional lower face of the valuated monomial support $\mathrm{msupp}_{\mathrm{ord}}(f)$ such that the projection $\pi(\mathcal{S}')$ is a subset of an edge of the Newton polytope. Then there exists a tropical point $\boldsymbol{\gamma} \in \mathrm{trop}(f) \cap \left(\frac{1}{N_1}\mathbb{Z}\right)^n$, where

$$N_1 = \frac{\mathrm{ll}(\pi(\mathcal{S}'))}{\mathrm{ll}(\mathcal{S}')}.$$

Proof. If $(\mathbf{i}_0, w_0), (\mathbf{i}_1, w_1) \in \mathbb{Z}^n \times \mathbb{Z}$ are the endpoints of the line segment $\operatorname{conv}(\mathcal{S}')$, let $\mathbf{a} = \mathbf{i}_1 - \mathbf{i}_0$, $h = w_1 - w_0$. By Remark 3.5, $\operatorname{ll}(\mathcal{S}')$ is equal to $\gcd(a_1, \ldots, a_n, h)$, which divides h. Thus, $\operatorname{ll}(\pi(\mathcal{S}'))$ divides

$$N_1 h = h \frac{\operatorname{ll}(\pi(\mathcal{S}'))}{\operatorname{ll}(\mathcal{S}')}.$$

Since $\operatorname{ll}(\pi(\mathcal{S}')) = \gcd(a_1, \ldots, a_n)$, there exists $\mathbf{n} \in \mathbb{Z}^n$ such that $\mathbf{a} \cdot \mathbf{n} = N_1 h$.

Let $\mathfrak{c}$ denote the normal cell corresponding to $\mathcal{S}'$. Its affine span is the $(n-1)$-dimensional hyperplane made up of the points $\mathbf{x} \in \mathbb{R}^n$ such that $(\mathbf{x}, 1)$ is orthogonal to $(\mathbf{a}, h)$, defined by the equation $\mathbf{a} \cdot \mathbf{x} + h = 0$. Since $\mathbf{a} \cdot (-\mathbf{n}/N_1) + h = 0$, the point $-\mathbf{n}/N_1$ lies on the hyperplane. By Proposition A.5, the normal cell $\mathfrak{c}$ contains arbitrarily large balls of this hyperplane. Hence, $\mathfrak{c}$ contains a point $\boldsymbol{\gamma}$ that differs from $-\mathbf{n}/N_1$ by an integer lattice vector. Therefore, $\boldsymbol{\gamma} \in \operatorname{trop}(f) \cap \left(\frac{1}{N_1}\mathbb{Z}\right)^n$. ☐

Let $\mathcal{A} = \operatorname{msupp}(f)$, and let $\tau\colon \mathbb{Z}^n \to \mathbb{Z}$ be a $\mathbb{Z}$-module homomorphism suich that $\max(\tau(\mathcal{A})) - \min(\tau(\mathcal{A})) = d$. Since the 1-skeleton of a convex polytope is connected, the Newton polytope $\operatorname{conv}(\mathcal{A})$ has an edge $\mathcal{S}$ such that $\tau(\mathcal{S}) \subset \mathbb{R}$ is nontrivial. Let $\mathcal{S}'$ be a nontrivial lower face of the valuated monomial support such that $\pi(\mathcal{S}') \subset \mathcal{S}$. Then Lemma 6.2 applies. We select the tropical point $\boldsymbol{\gamma}$ it defines.

The proof of Proposition 6.1 is completed by the following two lemmas.

Lemma 6.3. $\operatorname{ll}\big(\pi(\mathcal{S}')\big) \leq d$.

Proof. Since $\mathcal{S}$ and $\mathcal{S}'$ are both one-dimensional, τ is injective on $\mathcal{S}$. Since τ respects lattice points, $\operatorname{ll}(\mathcal{S}) \leq \operatorname{ll}(\tau(\mathcal{S}))$. Necessarily, $\tau(\mathcal{S})$ is a subset of the line segment $[\min(\tau(\mathcal{A})), \max(\tau(\mathcal{A}))]$, and so $\operatorname{ll}(\tau(\mathcal{S})) \leq d$. Finally, $\pi(\mathcal{S}') \subset \mathcal{S}$ implies that $\operatorname{ll}(\pi(\mathcal{S}')) \leq \operatorname{ll}(\mathcal{S})$. ☐

Lemma 6.4. $\operatorname{ll}\big(\mathcal{S}'\big) \geq \operatorname{p}\underline{\operatorname{mult}}_{\operatorname{trop}(f)}(\boldsymbol{\gamma})$.

Proof. By definition of tropical pseudo-multiplicity, $\operatorname{p}\underline{\operatorname{mult}}_{\operatorname{trop}(f)}(\boldsymbol{\gamma})$ is equal to $\operatorname{p}\underline{\operatorname{mult}}\big(\pi(\mathcal{S}')\big)$. Since $\pi(\mathcal{S}')$ is one-dimensional, its pseudo-multiplicity is equal to its lattice sublength, which divides $\operatorname{ll}(\mathcal{S}')$. ☐

If either of the inequalities given by 6.3 and 6.4 can be made strict for a particular f, then the bound of the Main Theorem is guaranteed to beat the naïve bound for this f.

Proposition 6.5. Let $f \in \mathbb{C}((t))[x_1, \ldots, x_n]$ be a Laurent-series polynomial whose monomial support has lattice width $d > 0$. Suppose no two parallel $(n-1)$-dimensional hyperplanes contain all vertices of the Newton polytope $\operatorname{New}(f)$. Then f has a zero $\bar{\mathbf{x}} \in \big(\mathbb{C}((t^{1/N}))^*\big)^n$ for some N strictly less than d.

Proof. Let $\tau\colon \mathbb{Z}^n \to \mathbb{Z}$ be as before. Let

$$m = \min\bigl(\tau(\mathrm{msupp}(f))\bigr), \qquad\qquad M = \max\bigl(\tau(\mathrm{msupp}(f))\bigr),$$

so that $d = M - m$. Then $\tau^{-1}(\{m\})$, $\tau^{-1}(\{M\})$ are the integer lattice points of two parallel hyperplanes in $\mathbb{R}^n$. By hypothesis, $\mathrm{New}(f)$ has a vertex $\mathbf{v} \in \mathbb{Z}^n$ such that $m < \tau(\mathbf{v}) < M$. This vertex has an adjacent edge $\mathcal{S}$ for which $\tau(\mathcal{S})$ has nonzero length. Necessarily, the length of $\tau(\mathcal{S})$ is strictly less than d, and so the lattice length of $\mathcal{S}$ is likewise. Now, there exists a one-dimensional lower face $\mathcal{S}'$ of the valuated monomial support $\mathrm{msupp}_{\mathrm{ord}}(f)$ such that $\pi(\mathcal{S}') \subset \mathcal{S}$. Clearly $\mathrm{ll}(\pi(\mathcal{S}')) \leq \mathrm{ll}(\mathcal{S})$, and hence $\mathrm{ll}(\pi(\mathcal{S}'))$ is strictly less than d.

By Lemma 6.2 there exists a point $\boldsymbol{\gamma} \in \mathrm{trop}(f) \cap \bigl(\frac{1}{N_1}\mathbb{Z}\bigr)^n$, where

$$N_1 = \frac{\mathrm{ll}(\pi(\mathcal{S}'))}{\mathrm{ll}(\mathcal{S}')}.$$

The proposition follows from Lemma 6.4 and the Main Theorem. □

For a polytope of dimension $n = 2$, the hypotheses of Proposition 6.5are equivalent to requiring that the Newton polytope is not a triangle or a trapezoid. For higher dimensions, polytopes not fulfilling the hypotheses include more complicated figures such as drums and pyramids. However, we may always say that if no two proper faces [i.e., faces that are proper subsets] of the polytope contain all the polytope's vertices, then it fulfills the hypotheses.

7 Proof of the Lazy Lemma

In this section, we prove Theorems 3.13 and 3.14.

Let $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ be an n-variable Laurent polynomial. We may write

$$f(\mathbf{x}) = \sum_{\mathbf{i} \in \mathcal{A}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}},$$

where $a_{\mathbf{i}} \in \mathbb{C}^*$ and $\mathcal{A} = \mathrm{msupp}(f)$. The *initial form* of f with respect to $\mathbf{u}$, denoted $\mathrm{in}_{\mathbf{u}}(f)(\mathbf{x}) \in \mathbb{C}[\mathbf{x}^{\pm 1}]$, is the Laurent polynomial given by

$$\mathrm{in}_{\mathbf{u}}(f)(\mathbf{x}) = \sum_{\mathbf{i} \in \mathrm{face}(\mathbf{u}, \mathcal{A})} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}.$$

The following easily verifiable fact is stated on p. 6 of [8].

Proposition 7.1. $\mathrm{in}_{\mathbf{u}}(f \cdot g) = \mathrm{in}_{\mathbf{u}}(f) \cdot \mathrm{in}_{\mathbf{u}}(g)$

Suppose f has irreducible factor p of multiplicity m. For zeros $\mathbf{a} \in (\mathbb{C}^*)^n$ of p, $\mathrm{mult}_f(\mathbf{a}) \geq m$, with equality outside a Zariski-closed subset of the hypersurface of p. Hence, the inferior multiplicity of f is the multiplicity, as a factor, of its least-multiplicity irreducible factor:

$$\underline{\mathrm{mult}}(f) = \min\{m > 0\colon p^m \mid f,\ p^{m+1} \nmid f,\ \text{some irreducible } p \in \mathbb{C}[\mathbf{x}^{\pm 1}]\}. \quad (2)$$

We may now prove Theorem 3.13, which is restated here for reference.

Theorem 3.13. Let $\mathcal{A}$ be a nontrivial monomial support. Then its inferior multiplicity is less than or equal to the inferior multiplicity of each of its nontrivial faces.

Proof. Suppose the inferior multiplicity of $\mathcal{A}$ is m. Then there exists a Laurent polynomial $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ with monomial support $\mathcal{A}$ and inferior multiplicity m. Let

$$f = p_1^{m_1} \cdots p_r^{m_r}$$

be the prime factorization of f. Since $\underline{\text{mult}}(f) = m$, we know that for all k, $m_k \geq m$.

Suppose $\mathcal{F} = \text{face}(\mathbf{u}, \mathcal{A})$ is a nontrivial face of $\mathcal{A}$. We observe that $\text{in}_{\mathbf{u}}(f)$ has monomial support $\mathcal{F}$. By Proposition 7.1,

$$\text{in}_{\mathbf{u}}(f) = \text{in}_{\mathbf{u}}(p_1)^{m_1} \cdots \text{in}_{\mathbf{u}}(p_r)^{m_r}.$$

Each prime factor p of $\text{in}_{\mathbf{u}}(f)$ must divide $\text{in}_{\mathbf{u}}(p_k)$, some k, and hence must have multiplicity at least $m_k \geq m$. Therefore, $\underline{\text{mult}}\big(\text{in}_{\mathbf{u}}(f)\big) \geq m$, and it follows that $\underline{\text{mult}}(\mathcal{F}) \geq m$. □

We now prove various versions of the Lazy Lemma.

Equation (2) suggests a definition of inferior multiplicity for elements of unique factorization domains in general that is coincides with the previous definition for the ring $\mathbb{C}[\mathbf{x}^{\pm 1}]$:

Definition 7.2. Let R be a unique factorization domain. For $f \in R$ neither zero nor a unit, the *inferior multiplicity* of f is the multiplicity of the least-multiplicity irreducible factor of f. If $f = p_1^{m_1} \cdots p_r^{m_r}$ is a factorization of f into distinct primes $p_1, \ldots, p_r$, then $\underline{\text{mult}}(f) = \min\{m_1, \ldots, m_r\}$.

With this definition, the following lemma is immediate.

Lemma 7.3. (Classic Lazy Lemma) Let $\phi\colon R \to R'$ be a homomorphism of unique factorization domains. For $f \in R$, if $\phi(f)$ is neither zero nor a unit, then $\underline{\text{mult}}(f) \leq \underline{\text{mult}}\big(\phi(f)\big)$.

We use monomial morphisms (Definition 2.6) to translate the Classic Lazy Lemma into affine maps of monomial supports.

Lemma 7.4. (Lazy Lemma version 1) Let $\mathcal{A} \subset \mathbb{Z}^n$ be a nontrivial monomial support, and let $\mu\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be an integer matrix injective on $\mathcal{A}$. Then

$$\underline{\text{mult}}(\mathcal{A}) \leq \underline{\text{mult}}\big(\mu(\mathcal{A})\big).$$

Proof. By the definition of inferior multiplicity for a monomial support,

$$\underline{\text{mult}}(\mathcal{A}) = \max_{f \in \text{msupp}^{-1}(\mathcal{A})} \big(\underline{\text{mult}}(f)\big).$$

By the Classic Lazy Lemma, $\underline{\text{mult}}(f) \leq \underline{\text{mult}}\big(\Phi_\mu(f)\big)$. Again by the definition of inferior multiplicity for a monomial support,

$$\begin{aligned} \underline{\text{mult}}\big(\Phi_\mu(f)\big) &\leq \underline{\text{mult}}\big(\text{msupp}(\Phi_\mu(f))\big) \\ &= \underline{\text{mult}}\big(\mu(\mathcal{A})\big). \end{aligned}$$

□

To go from this to a statement about linear maps over $\mathbb{Q}$, we need another lemma. Let d be a positive integer, and let $\mathbf{a}^d$ denote $(a_1^d, \ldots, a_n^d)$.

Lemma 7.5. Let $g(\mathbf{x}) \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ and $f(\mathbf{x}) = g(\mathbf{x}^d)$. Suppose $\mathbf{a} \in (\mathbb{C}^*)^n$ is a zero of f, so that $\mathbf{a}^d$ is a zero of g. Then $\operatorname{mult}_f(\mathbf{a}) = \operatorname{mult}_g(\mathbf{a}^d)$.

Proof. It suffices to consider the case

$$g(\mathbf{x}) = g_m(\mathbf{x} - \mathbf{a}^d),$$

where g_m is homogeneous of degree m and hence $m = \operatorname{mult}_g(\mathbf{a}^d)$. Let

$$\mathcal{I} = \{\mathbf{i} \in \mathbb{Z}^m \colon 1 \le i_1 \le \cdots \le i_m \le n\}.$$

Then $g_m(\mathbf{x}) = \sum_{\mathbf{i}\in\mathcal{I}} c_{\mathbf{i}} x_{i_1} \cdots x_{i_m}$, where $c_{\mathbf{i}} \in \mathbb{C}$ and for some $\mathbf{i} \in \mathcal{I}$, $c_{\mathbf{i}} \neq 0$. Thus,

$$\begin{aligned}
f(\mathbf{x}) &= g_m(\mathbf{x}^d - \mathbf{a}^d) \\
&= \sum_{\mathbf{i}\in\mathcal{I}} c_{\mathbf{i}} \prod_{\alpha=1}^{m} (x_{i_\alpha}^d - a_{i_\alpha}^d) \\
&= \sum_{\mathbf{i}\in\mathcal{I}} c_{\mathbf{i}} \prod_{\alpha=1}^{m} \prod_{j=1}^{d} (x_{i_\alpha} - \zeta^j a_{i_\alpha}) \\
&= \sum_{\mathbf{i}\in\mathcal{I}} c_{\mathbf{i}} \prod_{\alpha=1}^{m} (x_{i_\alpha} - a_{i_\alpha}) \prod_{j=1}^{d-1} \big((x_{i_\alpha} - a_{i_\alpha}) + (1-\zeta^j) a_{i_\alpha}\big) \\
&= \tilde{f}(\mathbf{x} - \mathbf{a}),
\end{aligned}$$

where ζ is a primitive d^{th} root of unity and

$$\tilde{f}(\mathbf{x}) = \sum_{\mathbf{i}\in\mathcal{I}} c_{\mathbf{i}} \prod_{\alpha=1}^{m} x_{i_\alpha} \prod_{j=1}^{d-1} \big(x_{i_\alpha} + (1-\zeta^j) a_{i_\alpha}\big).$$

Note that the least-degree nonzero homogeneous summand of $\tilde{f}$ is

$$f_m(\mathbf{x}) = \sum_{\mathbf{i}\in\mathcal{I}} c_{\mathbf{i}} \prod_{\alpha=1}^{m} x_{i_\alpha} \prod_{j=1}^{d-1} (1-\zeta^j) a_{i_\alpha},$$

which is homogeneous of degree m. Hence, $\operatorname{mult}_f(\mathbf{a}) = m$. ∎

Corollary 7.6. If $f(\mathbf{x}) = g(x_1^d, \ldots, x_n^d)$, then $\underline{\operatorname{mult}}(f) = \underline{\operatorname{mult}}(g)$.

Lemma 7.7. (Lazy Lemma version 2) Let $\mu \colon \mathbb{Q}^n \to \mathbb{Q}^n$ be a linear map, with $\mathcal{A} \subset \mathbb{Z}^n$ a nontrivial monomial support such that μ is injective on $\mathcal{A}$. If $\mu(\mathcal{A}) \subset \mathbb{Z}^n$, then $\underline{\operatorname{mult}}(\mathcal{A}) \le \underline{\operatorname{mult}}\big(\mu(\mathcal{A})\big)$.

Proof. Let $d \in \mathbb{Z}$ be the least common denominator for all the entries of the matrix of μ. By the Lazy Lemma version 1, $\underline{\text{mult}}(\mathcal{A}) \le \underline{\text{mult}}\big(d\mu(\mathcal{A})\big)$. Now,

$$\begin{aligned}\underline{\text{mult}}\big(d\mu(\mathcal{A})\big) &= \max_{g \in \text{msupp}^{-1}(\mu(\mathcal{A}))} \underline{\text{mult}}\big(g(x_1^d, \ldots, x_n^d)\big) \\ &= \max_{g \in \text{msupp}^{-1}(\mu(\mathcal{A}))} \underline{\text{mult}}(g) \\ &= \underline{\text{mult}}\big(\mu(\mathcal{A})\big). \qquad \square\end{aligned}$$

We now generalize the Lazy Lemma from linear to affine maps.

Lemma 7.8. Suppose $\mathcal{A} \subset \mathbb{Z}^n$ is a nontrivial monomial support and $\alpha\colon \mathbb{Q}^n \to \mathbb{Q}^n$ is an affine map, injective on $\mathcal{A}$, such that $\alpha(\mathcal{A}) \subset \mathbb{Z}^n$. Then $\underline{\text{mult}}(\mathcal{A}) \le \underline{\text{mult}}\big(\alpha(\mathcal{A})\big)$.

Proof. As in the proof of 7.7, we may scale by an integer factor. For an appropriate choice of integer factor, α becomes a linear map followed by translation by an integer vector. Since multiplication by a Laurent monomial does not alter the inferior multiplicity of a Laurent polynomial, translation by an integer vector does not alter the inferior multiplicity of a monomial support. $\square$

This gives the desired statement of the Lazy Lemma as a corollary.

Theorem 3.14. If α is an invertible affine map, then

$$\underline{\text{mult}}(\mathcal{A}) = \underline{\text{mult}}\big(\alpha(\mathcal{A})\big),$$

assuming $\mathcal{A}$ and $\alpha(\mathcal{A})$ are nontrivial monomial supports, i.e., are finite subsets of $\mathbb{Z}^n$ containing at least two points.

8 Setup for Proof of Theorem 5.3

In this section and the next, we present a constructive proof of Theorem 5.3. This construction is a generalization and strengthening of a construction on pp. 97-102 of [9], there used to prove that the field of Puiseux series is algebraically closed.

Definition 8.1. For a nonzero Puiseux series

$$s = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + a_3 t^{\alpha_3} + \cdots \in \mathbb{K}^*,$$

$a_k \in \mathbb{C}^*$, $\alpha_k < \alpha_{k+1}$, the *leading-term coefficient* $\text{lt}(s)$ is a_1. Similarly, for a nonzero polynomial

$$f(\mathbf{x}) = p_1(\mathbf{x}) t^{\alpha_1} + p_2(\mathbf{x}) t^{\alpha_2} + \cdots \in \mathbb{K}[\mathbf{x}],$$

$p_k \in \mathbb{C}[\mathbf{x}] \smallsetminus \{0\}$, the *leading-term polynomial* $\text{lt}(f)(\mathbf{x})$ is $p_1(\mathbf{x})$.

Let $f(\mathbf{x}) \in \mathbb{C}((t))[\mathbf{x}]$ be an n-variable Laurent-series polynomial. We may write

$$f(\mathbf{x}) = \sum_{(\mathbf{i},w)\in \mathrm{msupp}_{\mathrm{ord}}(f)} a_{\mathbf{i}} t^w \mathbf{x}^{\mathbf{i}}, \tag{3}$$

where $\mathrm{ord}(a_{\mathbf{i}}) = 0$. Let $\mathbf{x}t^{\boldsymbol{\gamma}}$ denote the componentwise product $(x_1 t^{\gamma_1}, \ldots, x_n t^{\gamma_n})$. A computation shows that

$$f(\mathbf{x}t^{\boldsymbol{\gamma}}) = \sum_{(\mathbf{i},w)\in \mathrm{msupp}_{\mathrm{ord}}(f)} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} t^{(\mathbf{i},w)\cdot(\boldsymbol{\gamma},1)}. \tag{4}$$

Let the n-variable polynomial $\phi_{\boldsymbol{\gamma}}(f) \in \mathbb{C}[\mathbf{x}]$ be defined by

$$\phi_{\boldsymbol{\gamma}}(f)(\mathbf{x}) = \mathrm{lt}\big(f(\mathbf{x}t^{\boldsymbol{\gamma}})\big),$$

so that

$$f(\mathbf{x}t^{\boldsymbol{\gamma}}) = \phi_{\boldsymbol{\gamma}}(f)(\mathbf{x})t^{\beta} + \text{h.o.t.} \tag{5}$$

for some $\beta \in \mathbb{Q}$, where h.o.t. stands for higher-order terms. From (4), we know

$$f(\mathbf{x}t^{\boldsymbol{\gamma}}) = \left(\sum_{(\mathbf{i},w)\in \mathcal{S}} \mathrm{lt}(a_{\mathbf{i}}) \mathbf{x}^{\mathbf{i}} t^{(\mathbf{i},w)\cdot(\boldsymbol{\gamma},1)} \right) + \text{h.o.t.},$$

where

$$\mathcal{S} = \mathrm{face}\big((\boldsymbol{\gamma},1), \mathrm{msupp}_{\mathrm{ord}}(f)\big). \tag{6}$$

We observe in (5) that

$$\beta = (\boldsymbol{\gamma},1)\cdot(\mathbf{i},w) \qquad \text{for all } (\mathbf{i},w) \in \mathcal{S}$$

and that

$$\phi_{\boldsymbol{\gamma}}(f)(\mathbf{x}) = \sum_{(\mathbf{i},w)\in \mathcal{S}} \mathrm{lt}(a_{\mathbf{i}}) \mathbf{x}^{\mathbf{i}}. \tag{7}$$

Suppose $\bar{\mathbf{x}} = \mathbf{c}t^{\boldsymbol{\gamma}} + \text{h.o.t.} \in \big(\mathbb{K}^*\big)^n$, where $\mathbf{c} \in (\mathbb{C}^*)^n$. We call $\bar{\mathbf{x}}$ a *lowest-order zero* of f if $\phi_{\boldsymbol{\gamma}}(f)(\mathbf{c}) = 0$. Note that

$$\begin{aligned} f(\bar{\mathbf{x}}) &= f(\mathbf{c}t^{\boldsymbol{\gamma}} + \text{h.o.t.}) \\ &= \phi_{\boldsymbol{\gamma}}(f)(\mathbf{c})t^{\beta} + \text{h.o.t.}; \end{aligned} \tag{8}$$

thus, $\mathrm{ord}\big(f(\bar{\mathbf{x}})\big) \geq \beta$, with strict inequality if and only if $\bar{\mathbf{x}}$ is a lowest-order zero.

Proposition 8.2. The Laurent-series polynomial $f(\mathbf{x})$ has a lowest-order zero of order $\boldsymbol{\gamma} \in \mathbb{Q}^n$ if and only if $\boldsymbol{\gamma} \in \mathrm{trop}(f)$.

Proof. A lowest-order zero $\mathbf{c}t^{\boldsymbol{\gamma}}+$h.o.t. of order $\boldsymbol{\gamma}$ exists if and only if the complex polynomial $\phi_{\boldsymbol{\gamma}}(f)$ has a zero $\mathbf{c} \in (\mathbb{C}^*)^n$. Such a point $\mathbf{c}$ exists if and only if $\phi_{\boldsymbol{\gamma}}(f)$ is not a monomial, i.e., has nontrivial monomial support. By (7), $\phi_{\boldsymbol{\gamma}}(f)$ has monomial support $\pi(\mathcal{S})$. Now, $\mathcal{S}$ is a lower face of the valuated monomial support $\mathrm{msupp}_{\mathrm{ord}}(f)$ and $\boldsymbol{\gamma}$ lies on the relative interior of the corresponding normal cell. Consequently, $\boldsymbol{\gamma}$ lies on the tropical hypersurface $\mathrm{trop}(f)$ if and only if $\mathcal{S}$ is nontrivial. □

The heart of the proof of the theorem is the following lemma, which is used to define the zero-series recursively.

Lemma 8.3. Given an n-variable Laurent-series polynomial $f \in \mathbb{C}((T))[\mathbf{x}]$ with a lowest-order zero $\mathbf{c}T^{\boldsymbol{\gamma}}$, where $\boldsymbol{\gamma} \in \operatorname{trop}(f) \cap \mathbb{Z}^n$, let $\widetilde{f} \in \mathbb{C}((T))[\mathbf{x}]$ be defined by

$$\widetilde{f}(\mathbf{x}) = f\big(T^{\boldsymbol{\gamma}}(\mathbf{c} + \mathbf{x})\big).$$

If $\widetilde{f}(\mathbf{0}) \neq 0$, then $\widetilde{f}$ has a lowest-order zero $\widetilde{\mathbf{c}}T^{\widetilde{\boldsymbol{\gamma}}}$ such that the components $\widetilde{\gamma_1}, \ldots, \widetilde{\gamma_n}$ of $\widetilde{\boldsymbol{\gamma}}$ are all strictly positive and $\widetilde{\boldsymbol{\gamma}} \in \big(\frac{1}{R}\mathbb{Z}\big)^n$, for some natural number

$$R \leq \frac{\operatorname{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})}{\operatorname{mult}_{\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})}(\widetilde{\mathbf{c}})}.$$

The remainder of this section is devoted to the proof of Lemma 8.3.

By assumption, $\widetilde{f}$ has nonzero constant term.

Lemma 8.4. No term of $\widetilde{f}$ with degree less than $\operatorname{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})$ has minimal order with respect to T, while at least one term with degree equal to $\operatorname{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})$ has minimal order with respect to T.

Proof. By definition of the leading-term polynomial,

$$\begin{aligned} \operatorname{lt}\big(f(\mathbf{x}T^{\boldsymbol{\gamma}})\big)(\mathbf{x}+\mathbf{c}) &= \operatorname{lt}\big(f((\mathbf{x}+\mathbf{c})T^{\boldsymbol{\gamma}})\big)(\mathbf{x}), \qquad \text{i.e.,} \\ \phi_{\boldsymbol{\gamma}}(f)(\mathbf{x}+\mathbf{c}) &= \operatorname{lt}(\widetilde{f})(\mathbf{x}); \end{aligned}$$

thus, $\operatorname{mult}_{\operatorname{lt}(\widetilde{f})}(\mathbf{0}) = \operatorname{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})$. Hence, no term of $\operatorname{lt}(\widetilde{f})$ has degree less than $\operatorname{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})$, but at least one term has degree equal to $\operatorname{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})$. ☐

Let $\mathbf{r}$ be a point of $\operatorname{msupp}_{\mathrm{ord}}(\widetilde{f})$ such that the w-coordinate is minimal and

$$\pi_1(\mathbf{r}) + \cdots + \pi_n(\mathbf{r}) = \operatorname{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c}); \tag{9}$$

the existence of such a point $\mathbf{r}$ is guaranteed by Lemma 8.4.

Lemma 8.5. There exists a real number $a > 0$ such that $(a, a, \ldots, a) \in \operatorname{trop}(\widetilde{f})$ and such that the corresponding subdivision cell contains $\mathbf{0}$.

Proof. Let $\mathcal{P} \subset \mathbb{R}^{n+1}$ be the valuated Newton polytope of $\widetilde{f}$. Since $\widetilde{f}$ is assumed to have nonzero constant term, $\mathcal{P}$ has a vertex $\mathbf{q}$ such that $\pi(\mathbf{q}) = \mathbf{0}$. Let I denote the interval $(0, \infty) \subset \mathbb{R}$, and let

$$\begin{aligned} &\sigma\colon I \to \mathbb{R}^{n+1} \qquad \text{be defined by} \\ &\sigma(s) = (s, s, \ldots, s, 1). \end{aligned}$$

By Proposition A.2, the set

$$\operatorname{face}\big(\sigma(I), \mathcal{P}\big) := \bigcup_{s \in I} \operatorname{face}(\sigma(s), \mathcal{P}) \subset \mathcal{P}$$

is path-connected. Since $\mathbf{r} \in \operatorname{face}\big(\sigma(\varepsilon), \mathcal{P}\big)$ and $\mathbf{q} \in \operatorname{face}\big(\sigma(\frac{1}{\varepsilon}), \mathcal{P}\big)$ for sufficiently small ε, there is a path within $\operatorname{face}\big(\sigma(I), \mathcal{P}\big)$ from $\mathbf{q}$ to $\mathbf{r}$. This path necessarily through the relative interior of a nontrivial face of $\mathcal{P}$ adjacent to $\mathbf{q}$. We call this face $\mathcal{F}$. Let $a \in I$ be such that $\mathcal{F} \subset \operatorname{face}\big(\sigma(a), \mathcal{P}\big)$. By definition of σ, $\pi\big(\operatorname{face}(\sigma(a), \mathcal{P})\big)$ is the subdivision cell corresponding to $(a, \ldots, a)$. Since $\mathcal{F}$ is nontrivial, the subdivision cell is nontrivial, and $(a, \ldots, a) \in \operatorname{trop}(\widetilde{f})$. ☐

We may now define $\widetilde{\boldsymbol{\gamma}}$ by

$$\widetilde{\boldsymbol{\gamma}} = (a, a, \ldots, a) \in \operatorname{trop}(\widetilde{f}). \tag{10}$$

For points $\mathbf{b} \in \mathbb{R}^{n+1}$, we write

$$\mathbf{b} = (\mathbf{i_b}, w_\mathbf{b}).$$

Thus $w_\mathbf{b} \in \mathbb{R}$ and $\mathbf{i_b} = \pi(\mathbf{b}) \in \mathbb{R}^n$. Let $\Sigma(\mathbf{i}) = i_1 + \cdots + i_n$.

Lemma 8.6. Every point $\mathbf{b}$ of

$$\mathcal{F} := \operatorname{face}\big((\widetilde{\boldsymbol{\gamma}}, 1), \mathcal{P}\big)$$

satisfies $\Sigma(\mathbf{i_b}) \leq \Sigma(\mathbf{i_r})$.

Proof. We know by definition of the face operator that $\mathbf{b} \cdot (\widetilde{\boldsymbol{\gamma}}, 1) \leq \mathbf{v} \cdot (\widetilde{\boldsymbol{\gamma}}, 1)$ for every $\mathbf{v} \in \mathcal{P}$. In particular, $(\mathbf{r} - \mathbf{b}) \cdot (\widetilde{\boldsymbol{\gamma}}, 1) \geq 0$. Since $\mathbf{r}$ has minimal w-coordinate, we see that $w_{\mathbf{r}-\mathbf{b}} \leq 0$. Thus,

$$\begin{aligned} \widetilde{\boldsymbol{\gamma}} \cdot \mathbf{i}_{\mathbf{r}-\mathbf{b}} &\geq \widetilde{\boldsymbol{\gamma}} \cdot \mathbf{i}_{\mathbf{r}-\mathbf{b}} + w_{\mathbf{r}-\mathbf{b}} \\ &= (\widetilde{\boldsymbol{\gamma}}, 1) \cdot (\mathbf{r} - \mathbf{b}) \\ &\geq 0. \end{aligned}$$

By definition of $\widetilde{\boldsymbol{\gamma}}$,

$$\begin{aligned} \widetilde{\boldsymbol{\gamma}} \cdot \mathbf{i}_{\mathbf{r}-\mathbf{b}} &= a\Sigma(\mathbf{i}_{\mathbf{r}-\mathbf{b}}) \\ &= a\big(\Sigma(\mathbf{i_r}) - \Sigma(\mathbf{i_b})\big). \end{aligned}$$

Since $a > 0$, this implies $\Sigma(\mathbf{i_r}) \geq \Sigma(\mathbf{i_b})$. ☐

Let $\mathcal{E}$ be an edge of $\mathcal{F} \subset \mathcal{P}$ with endpoints $\mathbf{q}$, $\mathbf{b}$. Let

$$\mathcal{E}' = \pi(\mathcal{E}) \cap \operatorname{msupp}(\widetilde{f});$$

thus, $\mathcal{E}'$ is a finite subset of $\mathbb{Z}^n$ with convex hull $\pi(\mathcal{E})$. Write

$$\mathcal{E} \cap \operatorname{msupp}_{\operatorname{ord}}(\widetilde{f}) = \{\mathbf{q}, \mathbf{b}_1, \ldots, \mathbf{b}_s\} \subset \mathbb{Z}^{n+1},$$

and select the indexing such that $\mathbf{b}_s$ is equal to $\mathbf{b}$, the endpoint of $\mathcal{E}$ opposite $\mathbf{q}$. Let

$$d = \gcd\big(\Sigma(\mathbf{i}_{\mathbf{b}_1}), \ldots, \Sigma(\mathbf{i}_{\mathbf{b}_s})\big).$$

Lemma 8.7. $a \in \frac{1}{d}\mathbb{Z}$.

Proof. Since $(\widetilde{\boldsymbol{\gamma}}, 1)$ is an inward normal for $\mathcal{E}$,

$$\begin{aligned} 0 &= (\widetilde{\boldsymbol{\gamma}}, 1) \cdot (\mathbf{b}_k - \mathbf{q}) \\ &= a\Sigma(\mathbf{i}_{\mathbf{b}_k - \mathbf{q}}) + w_{\mathbf{b}_k - \mathbf{q}}. \end{aligned}$$

Note that $\mathbf{i}_{\mathbf{b}_k - \mathbf{q}} = \mathbf{i}_{\mathbf{b}_k} - \mathbf{i}_{\mathbf{q}} = \mathbf{i}_{\mathbf{b}_k}$. Thus, $a\Sigma(\mathbf{i}_{\mathbf{b}_k}) \in \mathbb{Z}$. Since this applies for all k, $a \gcd\big(\Sigma(\mathbf{i}_{\mathbf{b}_1}), \ldots, \Sigma(\mathbf{i}_{\mathbf{b}_s})\big) \in \mathbb{Z}$, i.e., $a \in \frac{1}{d}\mathbb{Z}$. ◻

Lemma 8.8. Considered as a monomial support, $\mathcal{E}'$ has inferior multiplicity less than or equal to $\deg(\mathcal{E}')/d$.

Proof. This follows from the fact that $\deg(\mathcal{E}')/d$ is the lattice sublength of $\mathcal{E}'$. ◻

Proposition 8.9. $d \leq \dfrac{\Sigma(\mathbf{i}_{\mathbf{r}})}{\underline{\mathrm{mult}}\big(\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})\big)}$.

Proof. By (7), we know

$$\begin{aligned} \mathrm{msupp}\big(\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})\big) &= \pi\big(\mathrm{face}(\widetilde{\boldsymbol{\gamma}}, 1), \mathrm{msupp}_{\mathrm{ord}}(\widetilde{f})\big) \\ &= \pi(\mathcal{F}) \cap \mathrm{msupp}(\widetilde{f}). \end{aligned}$$

Since $\mathcal{E}'$ is a nontrivial face of $\pi(\mathcal{F}) \cap \mathrm{msupp}(\widetilde{f})$, it follows by Theorem 3.13 that

$$\begin{aligned} \underline{\mathrm{mult}}\big(\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})\big) &\leq \underline{\mathrm{mult}}(\mathcal{E}') \\ &\leq \frac{\deg(\mathcal{E}')}{d} \qquad \text{by Lemma 8.8.} \end{aligned}$$

By Lemma 8.6, we know that $\deg(\mathcal{E}') \leq \Sigma(\mathbf{i}_{\mathbf{r}})$. Hence,

$$\begin{aligned} \underline{\mathrm{mult}}\big(\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})\big) &\leq \frac{\Sigma(\mathbf{i}_{\mathbf{r}})}{d} \\ d &\leq \frac{\Sigma(\mathbf{i}_{\mathbf{r}})}{\underline{\mathrm{mult}}\big(\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})\big)}. \end{aligned}$$ ◻

By the original definition of inferior multiplicity of a polynomial, we know there exists a point $\widetilde{\mathbf{c}} \in (\mathbb{C}^*)^n$ that is a zero of $\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})$ such that

$$\mathrm{mult}_{\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})}(\widetilde{\mathbf{c}}) = \underline{\mathrm{mult}}\big(\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})\big).$$

By (9), $\Sigma(\mathbf{i}_{\mathbf{r}}) = \mathrm{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})$; hence, we find that

$$d \leq \frac{\mathrm{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})}{\mathrm{mult}_{\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})}(\widetilde{\mathbf{c}})}. \tag{11}$$

The quantities $\widetilde{\boldsymbol{\gamma}}$ and $\widetilde{\mathbf{c}}$ satisfy the following:

- The n-tuple $\widetilde{\mathbf{c}}t^{\widetilde{\boldsymbol{\gamma}}}$ is a lowest-order zero of $\widetilde{f}$: By definition, this is equivalent to $\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})(\widetilde{\mathbf{c}}) = 0$, which follows from the choice of $\widetilde{\mathbf{c}}$.

- The components $\widetilde{\gamma_1}, \ldots, \widetilde{\gamma_n}$ of $\widetilde{\boldsymbol{\gamma}}$ are all strictly positive: In (10), we defined $\widetilde{\gamma_1} = \cdots = \widetilde{\gamma_n} = a$. By Lemma 8.5, a is strictly positive.

- For some $R \leq \dfrac{\mathrm{mult}_{\phi_{\boldsymbol{\gamma}}(f)}(\mathbf{c})}{\mathrm{mult}_{\phi_{\widetilde{\boldsymbol{\gamma}}}(\widetilde{f})}(\widetilde{\mathbf{c}})}$, the point $\widetilde{\boldsymbol{\gamma}}$ lies in $\left(\frac{1}{R}\mathbb{Z}\right)^n$: Let $R = d$. By Lemma 8.7, $\widetilde{\boldsymbol{\gamma}} \in \left(\frac{1}{d}\mathbb{Z}\right)^n$. The inequality is precisely (11).

These are precisely the conclusions of Lemma 8.3.

9 Proof of Theorem 5.3

We now prove Theorem 5.3, which is restated here for convenience.

Theorem 5.3. Let $f(\mathbf{x}) \in \mathbb{C}((t))[\mathbf{x}]$ be an n-variable Laurent-series polynomial. Suppose we are given a point $\boldsymbol{\gamma}$ on $\mathrm{trop}(f) \cap \mathbb{Q}^n$. Let $N_1 \in \mathbb{N}$ be the least natural number such that $\boldsymbol{\gamma} \in \left(\frac{1}{N_1}\mathbb{Z}\right)^n$, and let $M = \underline{\mathrm{mult}}_{\mathrm{trop}(f)}(\boldsymbol{\gamma})$. Then for some natural number $N \leq N_1 M$, f has a zero $\bar{\mathbf{x}} \in \left(\mathbb{C}((t^{1/N}))^*\right)^n$ such that $\mathrm{ord}(\bar{\mathbf{x}}) = \boldsymbol{\gamma}$.

By definition of tropical inferior multiplicity $M = \underline{\mathrm{mult}}\big(\pi(\mathcal{S})\big)$, where $\mathcal{S}$ is as in (6). Now, by (7), $\mathrm{msupp}\big(\phi_{\boldsymbol{\gamma}}(f)\big) = \pi(\mathcal{S})$; hence, $\underline{\mathrm{mult}}\big(\phi_{\boldsymbol{\gamma}}(f)\big) \leq \underline{\mathrm{mult}}\big(\pi(\mathcal{S})\big) = M$. By definition of inferior multiplicity of a polynomial, this means $\phi_{\boldsymbol{\gamma}}(f)$ has a zero $\mathbf{c} \in (\mathbb{C}^*)^n$ of multiplicity at most M. Since $\mathbf{c}$ is a zero of $\phi_{\boldsymbol{\gamma}}(f)$, we know $\mathbf{c}t^{\boldsymbol{\gamma}} + \text{h.o.t.}$ is a lowest-order zero of f.

Using Lemma 8.3, we now recursively define

- $R_k, N_k \in \mathbb{N}$
- $f_k \in \mathbb{C}((t^{1/N_{k-1}}))[\mathbf{x}]$, or $\mathbb{C}((t))[\mathbf{x}]$ if $k = 1$
- $\boldsymbol{\gamma}_k \in \mathrm{trop}(f) \cap \left(\frac{1}{N_k}\mathbb{Z}\right)^n$
- $\mathbf{c}_k \in (\mathbb{C}^*)^n$ with $\phi_{\boldsymbol{\gamma}_k}(f_k)(\mathbf{c}_k) = 0$.

For $k = 1$, N_1 is as in the hypothesis of the theorem. We set $R_1 = N_1$, $f_1 = f$, $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}$, and $\mathbf{c}_1 = \mathbf{c}$.

For $k \geq 2$ we proceed as follows. If $\boldsymbol{\gamma}_{k-1} = \boldsymbol{\infty}$ then we set $\boldsymbol{\gamma}_k = \boldsymbol{\infty}$ and the remaining quantities become irrelevant. Otherwise, we define

- $f_k(\mathbf{x}) = f_{k-1}\big(t^{\boldsymbol{\gamma}_{k-1}}(\mathbf{c}_{k-1} + \mathbf{x})\big) \in \mathbb{C}((t^{1/N_{k-1}}))[\mathbf{x}]$.

If $f_k(\mathbf{0}) = 0$, then we set $\boldsymbol{\gamma}_k = \boldsymbol{\infty}$ and all other quantities become irrelevant. Otherwise, applying Lemma 8.3 with $T = t^{1/N_{k-1}}$, we see that there exist

- $R_k \in \mathbb{N}$

- $\boldsymbol{\gamma}_k \in \operatorname{trop}(f) \cap \left(\frac{1}{R_k N_{k-1}}\mathbb{Z}\right)^n$
- $\mathbf{c}_k \in (\mathbb{C}^*)^n$ with $\phi_{\boldsymbol{\gamma}_k}(f_k)(\mathbf{c}_k) = 0$

such that

$$R_k \leq \frac{\operatorname{mult}_{\phi_{\boldsymbol{\gamma}_{k-1}}(f_{k-1})}(\mathbf{c}_{k-1})}{\operatorname{mult}_{\phi_{\boldsymbol{\gamma}_k}(f_k)}(\mathbf{c}_k)}$$

and the components $(\boldsymbol{\gamma}_k)_1, \ldots, (\boldsymbol{\gamma}_k)_n$ of $\boldsymbol{\gamma}_k$ are all strictly positive. Finally, we let

- $N_k = N_{k-1}R_k$.

For each k,

$$\begin{aligned} N_k &= N_1 R_2 R_3 \cdots R_k \\ &\leq N_1 \cdot \frac{\operatorname{mult}_{\phi_{\boldsymbol{\gamma}_1}(f_1)}(\mathbf{c}_1)}{\operatorname{mult}_{\phi_{\boldsymbol{\gamma}_2}(f_2)}(\mathbf{c}_2)} \cdots \frac{\operatorname{mult}_{\phi_{\boldsymbol{\gamma}_{k-1}}(f_{k-1})}(\mathbf{c}_{k-1})}{\operatorname{mult}_{\phi_{\boldsymbol{\gamma}_k}(f_k)}(\mathbf{c}_k)} \\ &\leq N_1 \cdot \operatorname{mult}_{\phi_{\boldsymbol{\gamma}_1}(f_1)}(\mathbf{c}_1) \\ &\leq N_1 M. \end{aligned}$$

Thus, there exists $N \leq N_1 M$ such that for each k, N_k divides N. For every $k \geq 1$, $\boldsymbol{\gamma}_k \in (\frac{1}{N}\mathbb{Z})^n$ with strictly positive components, and we may define the series

$$\bar{\mathbf{x}}_k = \mathbf{c}_k t^{\boldsymbol{\gamma}_k} + \mathbf{c}_{k+1} t^{\boldsymbol{\gamma}_k + \boldsymbol{\gamma}_{k+1}} + \mathbf{c}_{k+2} t^{\boldsymbol{\gamma}_k + \boldsymbol{\gamma}_{k+1} + \boldsymbol{\gamma}_{k+2}} + \cdots \in \mathbb{C}((t^{1/N}))^n,$$

adopting the convention that $t^{\boldsymbol{\infty}} = \mathbf{0}$. Thus, for $k \geq 2$,

$$\bar{\mathbf{x}}_k = t^{\boldsymbol{\gamma}_k}(\mathbf{c}_k + \bar{\mathbf{x}}_{k+1}),$$

and for all $k \geq 1$,

$$f_{k+1}(\bar{\mathbf{x}}_{k+1}) = f_k(\bar{\mathbf{x}}_k). \tag{12}$$

Since for each $k > 1$, every component of $\boldsymbol{\gamma}_k$ is strictly positive, we know that for all $k \geq 1$, the order of $\bar{\mathbf{x}}_k$ is $\boldsymbol{\gamma}_k$.

It remains to show that $\bar{\mathbf{x}}_1$ is actually a zero of f. If the $\boldsymbol{\gamma}_k$ are not all finite, or equivalently if the $\bar{\mathbf{x}}_k$ are finite sums, then this is relatively easy. Let ℓ be the first index for which $\boldsymbol{\gamma}_\ell = \boldsymbol{\infty}$. Then we have $\bar{\mathbf{x}}_\ell = \mathbf{0}$ and by (12), $f_1(\bar{\mathbf{x}}_1) = f_\ell(\bar{\mathbf{x}}_\ell) = f_\ell(\mathbf{0})$. Since $f_\ell(\mathbf{0}) = 0$ was precisely the criterion for setting $\boldsymbol{\gamma}_\ell = \boldsymbol{\infty}$, $\bar{\mathbf{x}}_1$ is the desired zero of $f = f_1$.

If all the $\boldsymbol{\gamma}_k$ are finite, a little more work is required. For all k, let

$$\begin{aligned} \mathcal{S}_k &= \operatorname{face}\big((\boldsymbol{\gamma}_k, 1), \operatorname{msupp}_{\mathrm{ord}}(f_k)\big) \\ \beta_k &= (\mathbf{i}, w) \cdot (\boldsymbol{\gamma}_k, 1) \qquad \text{for all } (\mathbf{i}, w) \in \mathcal{S}_k. \end{aligned}$$

Since $\boldsymbol{\gamma}_{k+1}$ lies on the tropical hypersurface, we know the corresponding face $\mathcal{S}_{k+1}$ is nontrivial. Hence, there is a point $(\mathbf{i}, w) \in \mathcal{S}_{k+1}$ such that $\mathbf{i} \neq \mathbf{0}$. Since $\mathbf{i}$

has all components nonnegative and $\boldsymbol{\gamma}_{k+1}$ has all components strictly positive, $\mathbf{i} \cdot \boldsymbol{\gamma}_{k+1} > 0$. Therefore,

$$\begin{aligned}\beta_{k+1} &= (\mathbf{i}, w) \cdot (\boldsymbol{\gamma}_{k+1}, 1) \\ &= \mathbf{i} \cdot \boldsymbol{\gamma}_{k+1} + w \\ &> w.\end{aligned}$$

Let $\mathbf{y} = t^{\boldsymbol{\gamma}_k}(\mathbf{c}_k + \mathbf{x})$, so that

$$\begin{aligned}f_{k+1}(\mathbf{x}) &= f_k(\mathbf{y}) \\ &= \phi_{\boldsymbol{\gamma}_k}(f_k)\big(\mathrm{lt}(\mathbf{y})\big)t^{\beta_k} + \text{h.o.t.}\end{aligned}$$

by (8). Thus, $w \geq \mathrm{ord}\big(f_{k+1}(\mathbf{x})\big) = \mathrm{ord}\big(f_k(\mathbf{y})\big) = \beta_k$. Hence, $\beta_{k+1} > \beta_k$. Since for all k, $\beta_k \in \frac{1}{N_k}\mathbb{Z} \subset \frac{1}{N}\mathbb{Z}$, the β_k become arbitrarily large. By (8),

$$f_k(\bar{\mathbf{x}}_k) = \phi_{\boldsymbol{\gamma}_k}(f_k)(\mathbf{c}_k)t^{\beta_k} + \text{h.o.t.},$$

and so $\mathrm{ord}\big(f_k(\bar{\mathbf{x}}_k)\big) > \beta_k$. By (12), $f_k(\bar{\mathbf{x}}_k) = f_{k+1}(\bar{\mathbf{x}}_{k+1})$. Hence, for all k, $\mathrm{ord}\big(f_k(\bar{\mathbf{x}}_k)\big) = \infty$, i.e., $f_k(\bar{\mathbf{x}}_k) = 0$. In particular, $\bar{\mathbf{x}}_1$ is the desired zero of $f = f_1$.

A Appendix

This section consists largely of the proofs for propositions concerning discrete geometry that are logically necessary but conceptually somewhat tangential to the paper.

Proposition A.1. Given a convex polytope $\mathcal{P}$ and a vector $\mathbf{u} \in \mathbb{R}^n$, there exists $\varepsilon > 0$ such that $\|\mathbf{w}\| < \varepsilon$ implies

$$\mathrm{face}(\mathbf{u} + \mathbf{w}, \mathcal{P}) = \mathrm{face}\big(\mathbf{w}, \mathrm{face}(\mathbf{u}, \mathcal{P})\big).$$

Proof. We may adapt the arguments used to prove a closely related statement on p. 33 of [4]. Another closely related proposition is stated without proof in [8], p. 10. □

For subsets $\mathcal{S}$ of $\mathbb{R}^n$, let $\mathrm{face}(\mathcal{S}, \mathcal{P})$ denote $\bigcup_{\mathbf{n} \in \mathcal{S}} \mathrm{face}(\mathbf{n}, \mathcal{P})$.

Proposition A.2. If $\mathcal{P} \subset \mathbb{R}^n$ is a convex polytope and $\mathcal{S} \subset \mathbb{R}^n$ is any connected set, then $\mathrm{face}(\mathcal{S}, \mathcal{P})$ is path-connected.

Proof. For each $\mathbf{n} \in \mathcal{S}$, we know $\mathrm{face}(\mathbf{n}, \mathcal{P})$ is path-connected (since convex). We may partition $\mathcal{S}$ into equivalence classes, where two vectors are considered equivalent if their faces belong to the same path component of $\mathrm{face}(\mathcal{S}, \mathcal{P})$. By Proposition A.1, there is about each vector an open ball of equivalent points. Hence, each equivalence class is open. Since $\mathcal{S}$ is connected, there is only one equivalence class. □

Lemma A.3. Let $\mathcal{P}$ be a convex polytope in $\mathbb{R}^{n+1}$ with lower face $\mathcal{S}$, and let $(\boldsymbol{\gamma}, 1)$ be an inward normal to $\mathcal{S}$. For a given $\mathbf{v} \in \mathbb{R}^n$, suppose $\pi(\mathcal{S}) \subset \operatorname{face}\big(\mathbf{v}, \pi(\mathcal{P})\big)$. Then $\mathcal{S} \subset \operatorname{face}\big((\boldsymbol{\gamma} + \mathbf{v}, 1), \mathcal{P}\big)$.

Proof. Let $(\mathbf{i}, w)$ be a point of $\mathcal{S}$. We are given that among the points of $\mathcal{P}$, the point $(\mathbf{i}, w)$ minimizes both $(\mathbf{i}, w) \cdot (\boldsymbol{\gamma}, 1)$ and $\mathbf{i} \cdot \mathbf{v}$. Thus, $(\mathbf{i}, w)$ minimizes

$$(\mathbf{i}, w) \cdot (\boldsymbol{\gamma}, 1) + \mathbf{i} \cdot \mathbf{v} = (\mathbf{i}, w) \cdot (\boldsymbol{\gamma} + \mathbf{v}, 1).$$

Hence, $(\mathbf{i}, w)$ is a point of $\operatorname{face}\big((\boldsymbol{\gamma} + \mathbf{v}, 1), \mathcal{P}\big)$. □

Assume we have a Laurent-series polynomial $f \in \mathbb{C}((t))[x_1, \ldots, x_n]$. Let $\mathcal{P} \subset \mathbb{R}^{n+1}$ denote the valuated Newton polytope of f.

Lemma A.4. Suppose $\mathcal{S}$ is a lower face of $\mathcal{P}$ with corresponding normal cell $\mathfrak{c} \in \Delta_N(f)$. If $\mathbf{v} \in \mathbb{R}^n$ is an inward normal to a face $\mathcal{F}$ of the (non-valuated) Newton polytope $\operatorname{New}(f)$, and $\pi(\mathcal{S}) \subset \mathcal{F}$, then $\mathfrak{c} + \mathbf{v} \subset \mathfrak{c}$.

Proof. Suppose $\boldsymbol{\gamma}$ is a point of the normal cell $\mathfrak{c}$. By the definition of normal cell, $(\boldsymbol{\gamma}, 1)$ is an inward normal to $\mathcal{P}$. By the definition of inward normal, we find that $\mathcal{F}$ is a subset of $\operatorname{face}\big(\mathbf{v}, \operatorname{New}(f)\big)$; hence, $\pi(\mathcal{S}) \subset \operatorname{face}\big(\mathbf{v}, \operatorname{New}(f)\big)$. This implies by Lemma A.3 that

$$\mathcal{S} \subset \operatorname{face}\big((\boldsymbol{\gamma} + \mathbf{v}, 1), \mathcal{P}\big),$$

i.e., $\boldsymbol{\gamma} + \mathbf{v}$ is a point of the normal cell $\mathfrak{c}$. □

Proposition A.5. Suppose $\mathcal{S}$ is a lower edge of $\mathcal{P}$ such that the projection $\pi(\mathcal{S})$ is a subset of an edge of the Newton polytope of f. Then the corresponding normal cell $\mathfrak{c} \in \Delta_N(f)$ contains arbitrarily large balls of its affine span.

Proof. Since $\pi(\mathcal{S})$ is a one-dimensional subset of an edge of a convex polytope that lies in $\mathbb{R}^n$, its inward normals form a convex set (the normal cone at that edge) of dimension $n-1$. Thus, there exists an $(n-1)$-dimensional ball B of inward normals. By multiplying by an arbitrarily large positive scalar if necessary, we may assure that B is arbitrarily large. The Minkowski sum $B + \mathfrak{c}$ clearly contains an $(n-1)$-dimensional ball of the desired size. By Lemma A.4, $B + \mathfrak{c}$ is a subset of $\mathfrak{c}$. Since the affine span of $\mathfrak{c}$ is an $(n-1)$-dimensional hyperplane, this completes the proof. □

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