

The Lefschetz Principle through model theory

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This document gives some illustration of using model theory to state and apply the Lefschetz Principle. Although no references are given, nothing in here should be taken to be original. For a discussion in which Brian Conrad gives specific references for alternative (and probably stronger) versions of the Lefschetz Principle based on flatness rather than model theory, see the remarks after <http://mathoverflow.net/questions/44758#44758>.

The “baby” version of the Lefschetz principle through model theory is the following:

Theorem 1. Let \mathbb{k} be an algebraically closed field, and \mathbb{K} an algebraically closed extension of \mathbb{k} . Let P be any first-order statement in the language of fields. Then P is true for \mathbb{k} iff P is true for \mathbb{K} .

This is a “baby” version because very few statements of interest in algebraic geometry can be expressed by a single first-order statement in the language of fields. For instance, one cannot directly express the statement that “ \mathbb{k} is algebraically closed.” This is because the statement we care about involves quantifying over polynomials, rather than just over elements of the field.

However, this method becomes surprisingly powerful if one considers statements that are equivalent to infinite collections $\{P_i\}$ of first-order statements. Let S be the statement we care about (which cannot be expressed as a first-order statement about the base field). Then we might be able to do something like

$$\begin{aligned} S \text{ holds over } \mathbb{k} &\iff \text{for each } i, P_i \text{ holds over } \mathbb{k} \\ &\iff \text{for each } i, P_i \text{ holds over } \mathbb{K} \\ &\iff S \text{ holds over } \mathbb{K}. \end{aligned}$$

As an example of how a statement might be “divided” into infinitely many first-order statements, consider the example mentioned earlier, the statement that “ \mathbb{k} is algebraically closed.” While this statement (call it S) cannot be expressed in the first-order language, the statement “Every polynomial of degree $\leq n$ has a root” (call this S_n) *can* be expressed in the first-order language. For instance, if $n = 3$, we write

$$\forall a_0 \forall a_1 \forall a_2 \forall a_3 \exists x (a_0 + a_1x + a_2xx + a_3xxx = 0).$$

Then S holds iff for all n , S_n holds. Consequently, we can use Theorem 1 to deduce that “ \mathbb{k} is algebraically closed iff \mathbb{K} is algebraically closed.” This is hardly impressive, since \mathbb{k} and \mathbb{K} are both algebraically closed by hypothesis. However, I should note that the reduction of S to $\{S_n\}$ was in fact crucial to the proof of Theorem 1.

For a somewhat more impressive example, I will use Theorem 1 to deduce the following statement:

Proposition 2. Let A be a finitely generated \mathbb{k} -algebra. Then A is reduced iff $A \otimes_{\mathbb{k}} \mathbb{K}$ is reduced.

For the first step, we note that A can be expressed as

$$A = \mathbb{k}[x_1, \dots, x_n]/(f_1, \dots, f_k).$$

We will also assume that f_1, \dots, f_k form a Gröbner basis, with the following key consequence:

$$\begin{aligned} \text{There exists a function } \phi: \mathbb{N} \rightarrow \mathbb{N} \text{ such that, if } f \in I \text{ and } \deg(f) \leq d, \\ \text{then } f = \sum_i a_i f_i \text{ for some } a_i \text{ of degree } \leq \phi(n). \end{aligned} \tag{1}$$

In other words, once we have bounded the degree of f , we can test whether $f \in I$ by testing whether it can be written as a linear combination of the f_i , in which the coefficients have bounded degree. This is significant since we can quantify over polynomials of bounded degree (by quantifying over their coefficients), but we can never quantify over all polynomials.

The statement S that “ A is reduced” is, by definition, equivalent to the statement that “for all $f \in \mathbb{k}[x_1, \dots, x_n]$ and all $m \in \mathbb{N}$, if $f^m \in I$, then $f \in I$.” A first potential obstacle to overcome is that we cannot quantify over integers. However, this is easy to patch: since A is Noetherian, every ideal of nilpotents is itself nilpotent. Thus, there exists m_0 such that $f \in A$ is nilpotent iff $f^{m_0} = 0$. Consequently, S is equivalent to the statement

$$\text{For all } f \in \mathbb{k}[x_1, \dots, x_n], \text{ if } f^{m_0} \in I, \text{ then } f \in I.$$

Remark. Note: we have not actually constructed any statement of this form, since we have no idea what m_0 is. However, for the purpose of applying Theorem 1, we don’t need to explicitly construct our statements equivalent to S ; we need only prove that they exist.

Remark. I believe that the technique below could also be used, in principle, to “quantify” over infinitely many m by dividing the statement into infinitely many weaker statements. If I am correct here, then the reduction to a single m_0 was a convenient shortcut, but not strictly necessary.

Our next difficulty is that we cannot quantify over all polynomials f . However, by quantifying over coefficients, we can quantify over all polynomials of degree $\leq d$. Thus, let S_d be the statement

For all $f \in \mathbb{k}[x_1, \dots, x_n]$ of degree $\leq d$, if $f^{m_0} \in I$, then $f \in I$.

Now, using (1), we may rewrite S_d as the statement

For all $f \in \mathbb{k}[x_1, \dots, x_n]$ of degree $\leq d$, if there exist a_i of degree $\leq \phi(m_0 d)$ such that $f^{m_0} = \sum_i a_i f_i$, then there exist b_i of degree $\leq \phi(d)$ such that $f = \sum_i b_i f_i$.

This, at last, is a statement that can be expressed in the first-order language of fields (by quantifying over the coefficients of f , the a_i , and the b_i . Note that the f_i are fixed, so we don't need to quantify over their coefficients.) By Theorem 1, for each d , S_d holds over \mathbb{k} iff S_d holds over \mathbb{K} . Since A is reduced iff S_d holds over \mathbb{k} for every d , and

$$A \otimes_{\mathbb{k}} \mathbb{K} = \mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_k)$$

is reduced iff S_d holds over \mathbb{K} for every d , we conclude that A is reduced iff $A \otimes_{\mathbb{k}} \mathbb{K}$ is reduced.

Remark. I have seen at least one paper that formalizes the method above by using a language that allows “infinite conjunctions” and the like. (The key characteristic is that all the infinite logical operations must occur *outside* all the quantifiers.) While it is nice to have a single stronger statement rather than a vague, example-based technique for strengthening a weaker statement, I am not convinced that the stronger statement is actually useful in trying to analyze specific statements about algebraic geometry.

The following exercise is considerably more involved than the example done in the text, but offers a nice example of the model theory interacting with the geometry.

Exercise 1. Use the statements above to prove the following statement: If A is a finitely generated, integrally closed domain over \mathbb{k} , then the natural group homomorphism $A^\times \rightarrow (A \otimes_{\mathbb{k}} \mathbb{K})^\times$ is an isomorphism. (Hint: First, use purely algebro-geometric techniques to show that the groups in question are finitely generated. The same techniques can be used to obtain the necessary bounds on degree.)