Elementary proof of generic freeness

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The following proof was taught to me by Madhav Nori, who learned it from Pavaman Murthy, who read it in Matsumura's book *Commutative Ring Theory* [Mat89]. Having reviewed Matsumura's version, I have to say that although it is recognizably the same proof, I think that the exposition has improved significantly in the course of being passed down through several other mathematicians.

Lemma. Let A be a ring (commutative, with unit), and M an A-module that is an increasing union of submodules M_n. (Assume $M_0 = 0$.) Suppose that for all n, the quotient M_{n+1}/M_n is free. Then M is free.

Proof. First, we inductively construct compatible isomorphisms

$$\phi_n \colon \bigoplus_{i=1}^{n} M_i/M_{i-1} \to M_n.$$

The base case $n = 1$ is obvious.

For the induction step, consider the short exact sequence

$$0 \to M_{n-1} \to M_n \to M_n/M_{n-1} \to 0.$$

Since M_n/M_{n-1} is free, the sequence splits. Thus, there is a (non-natural) isomorphism

$$M_{n-1} \oplus M_n/M_{n-1} \to M_n.$$

Applying the isomorphism ϕ_{n-1} that exists by inductive hypothesis, we obtain the desired ϕ_n.

Since the ϕ_n are compatible, we may take

$$\phi := \varinjlim \phi_n \colon \bigoplus_{i=1}^{\infty} M_i/M_{i-1} \to M.$$

Since the $\varinjlim$ is a functor, it takes isomorphisms to isomorphisms. Since $\{\phi_n\}$ is an isomorphism of directed systems, the direct limit ϕ is an isomorphism. (Alternately, it is easy to verify injectivity and surjectivity by looking at elements.) □

Remark. The argument above actually shows that if the M_{n+1}/M_n are all projective, then M is (non-naturally) isomorphic to their direct sum.

We now progress to the main result. Throughout the following, let A be a fixed Noetherian integral domain. Call an A-algebra B "nice" if for every finite B-module M, there exists nonzero $f \in A$ such that M_f is free as an A_f-module.

Theorem. (Generic freeness) Every finitely generated A-algebra is nice.

Proof. We prove this theorem via a series of exercises.

Exercise 1. Show that A is nice as an A-algebra.

Exercise 2. Reduce to the following statement: If B is a nice Noetherian A-algebra, so is the polynomial ring $B[T]$. (Hint: induct on the number of generators of B as an A-algebra.)

Assume B is a nice A-algebra, and let M be a finite $B[T]$-module. Let $S \subset M$ be a finite set that generates M as a $B[T]$-module. Let M_1 be the sub-B-module of M generated by S. Inductively define

$$M_{n+1} = M_n + TM_n,$$

a sub-B-module of M. Observe that, as a B-module, M is the increasing union of the M_n.

Exercise 3. For $n \gg 0$, the B-module M_n/M_{n-1} is isomorphic to M_{n+1}/M_n. (Hint: Note that for all n, T defines a surjective B-module morphism $M_n/M_{n-1} \to M_{n+1}/M_n$. Apply the ascending chain condition on M_1.)

Exercise 4. There exists nonzero $f \in A$ such that $(M_{n+1}/M_n)_f$ is free as an A_f-module, for all n. (As n ranges over the natural numbers, M_{n+1}/M_n only hits finitely many isomorphism classes.)

Applying the Lemma, we see that M_f is free as an A_f-module. □

References

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.