1 The Local-to-Global Lemma

In the world of advanced mathematics, we are often interested in comparing the local properties of a space to its global properties. Connectedness is one of our most important tools in doing this. Often, it is easier to prove that a property holds in a neighborhood of each point than to prove that it holds for the entire space. If the space is connected, then using the following lemma, we can sometimes deduce that a property holds globally, simply from the fact that it holds in a neighborhood of every point.

Local-to-Global Lemma. Let X be connected. Suppose that we have an equivalence relation $\sim$ on X such that every point has a neighborhood of equivalent points. Then there is only one equivalence class, i.e., all points of X are equivalent.

Proof. Let $X = \bigcup_\alpha X_\alpha$ be the partition of X into equivalence classes X_α. [Thus, X is the disjoint union of the X_α, and if $x \in X_\alpha$, then $y \in X_\alpha \iff x \sim y$.] By hypothesis, for all $x \in X_\alpha$, there exists an open set U such that $x \in U$ and every element of U is equivalent to x; i.e., $x \in U \subset X_\alpha$. Thus, every equivalence class X_α is open.

Suppose there is more than one equivalence class. Let $U = X_\alpha$ be an equivalence class, and $V = \bigcup_{\beta \neq \alpha} X_\beta$. Then U, V form a separation, contradicting the connectedness of X. Hence, there is only one equivalence class. $\square$

Here are two important examples of how this lemma can be used.

Proposition 1. Let X be a topological space, Y a set. Suppose $f\colon X \to Y$ is locally constant, i.e., every $x \in X$ has a neighborhood U on which $f \restriction_U$ is constant. If X is connected, then f is constant.

Proof. Let $\sim$ be the relation on X defined by

$$x \sim y \iff f(x) = f(y).$$

Clearly, $\sim$ is an equivalence relation. By hypothesis, every point of X has a neighborhood of equivalent points. By the local-to-global lemma, $x \sim y$ for every $x, y \in X$; i.e., $f(x) = f(y)$ for every $x, y \in X$; i.e., f is constant. $\square$

For the proof of the next proposition, we will need to recall how to compose paths, which was discussed in the last lecture but was not included on that handout. Suppose ϕ is a path from a to b, and ψ a path from b to c. Define a path

$$\phi \cdot \psi \colon I \to X$$

$$(\phi \cdot \psi)(t) = \begin{cases} \phi(2t), & t \in [0, \frac{1}{2}] \\ \psi(2t - 1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

By the pasting lemma, $\phi \cdot \psi$ is continuous. Thus, it is a path in X from a to c. Intuitively, we may think of it as the path obtained by following ϕ from a to b, and then following ψ from b to c.

Proposition 2. Let $X \subset \mathbb{R}^n$ be open. If X is connected, then X is path connected.

Proof. Define $a \sim b$ to mean "there is a path from a to b." We claim that $\sim$ is an equivalence relation.

Reflexive: For every $a \in X$, the constant path $I \to \{a\}$ is a path from a to itself; hence, $a \sim a$.

Symmetric: Suppose $\phi \colon I \to X$ is a path from a to b. Then $\psi(t) = \phi(1 - t)$ is a path from b to a. Thus, $a \sim b \Longrightarrow b \sim a$.

Transitive: Suppose $a \sim b$ and $b \sim c$. Let ϕ be a path from a to b, and ψ a path from b to c. Then the composition of paths $\phi \cdot \psi$ is a path from a to c. Hence, $a \sim c$.

Thus, $\sim$ is an equivalence relation, as claimed. Let $x \in X$. Since X is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset X$. Then $B_\varepsilon(x)$ is path-connected, hence all points of $B_\varepsilon(x)$ are equivalent. By the local-to-global lemma, all points of X are equivalent, i.e., X is path-connected. $\square$

It is an interesting bit of culture to see a more sophistocated use of the local-to-global lemma. The following comes from page 24 of Milnor's book *Topology from the Differentiable Viewpoint*:

> Now consider a connected manifold N. Call two points of N "iso-topic" if there exists a smooth isotopy carrying one to the other. This is clearly an equivalence relation. If y is an interior point, then it has a neighborhood diffeomorphic to $\mathbb{R}^n$; hence the above argument shows that every point sufficiently close to y is "isotopic" to y. In other words, each "isotopy class" of points in the interior of N is an open set, and the interior of N is partitioned into disjoint open isotopy classes. But the interior of N is connected; hence there can be only one such isotopy class.

I do not expect you to understand all the words Milnor uses, but you should be able to recognize that the local-to-global lemma is being used (in fact, is being proved on the fly).

2   Local Path Connectedness

The statement that open, connected subsets of $\mathbb{R}^n$ are path connected may be generalized with the following definition.

Definition 3. A space X is *locally path connected* if it has a basis of path connected subsets, i.e., if for every point x and every neighborhood U of x, there is a path connected open set V such that $x \in V \subset U$. (Intuitively, every point has arbitrarily small connected neighborhoods.)

Note that every open subset of $\mathbb{R}^n$ is locally path connected, since the open balls form a path connected basis.

Proposition 4. If a space is connected and locally path connected, then it is path connected.

Proof. This follows from the local-to-global lemma, using the same equivalence relation as for Proposition 2: $a \sim b$ if there is a path from a to b. □

The property of local path connectedness is important in algebraic topology. Note that a space can be path connected without being locally path connected.

Example. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Let $X \subset \mathbb{R}^2$ be the union of the set of vertical lines $(K \cup \{0\}) \times \mathbb{R}$ and the horizontal line $\mathbb{R} \times \{0\}$.
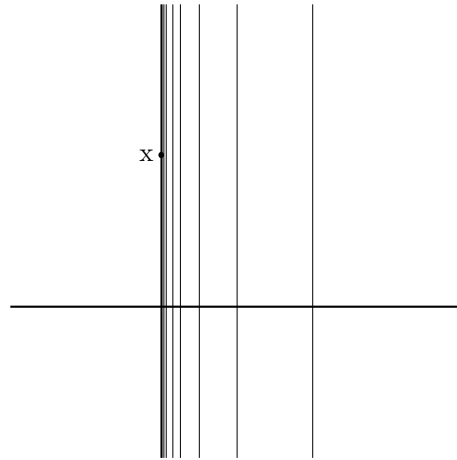
Then X is clearly path connected. However, consider the point $x = (0,1)$ on the leftmost vertical line, and the neighborhood $U = X \smallsetminus (\mathbb{R} \times \{0\})$ obtained by deleting the horizontal line. Within U, no neighborhood of x can be connected, much less path connected.

(Careful proof: any neighborhood V of x contains a product of intervals $((a,b) \times (c,d)) \cap U$ about $x = \langle 0, 1 \rangle$, where $a < 0 < b$ and $c < 1 < d$. Thus, for sufficiently large n, the point $\langle \frac{1}{n}, 1 \rangle$ lies in V. Then for any $q \in (\frac{1}{n+1}, \frac{1}{n})$, we have a separation of V into $V \cap ((-\infty, q) \times \mathbb{R})$ and $V \cap ((q, \infty) \times \mathbb{R})$.)

3   Path Components and Components

We have seen that the relation "$a \sim b$ if there is a path from a to b" is an equivalence relation on any topological space. Even (perhaps especially) for spaces that are not path connected, it can be interesting to study the equivalence classes.

Definition 5. The equivalence classes of the relation described above, for a space X, are called the *path components* of X.

By definition, there is a path from a to b if and only if a and b lie in the same path component of X. X is the disjoint union of its path components. If C is a path component, then there is a path between any two points of C, i.e., C is path connected. Moreover, there is not any path from a point in C to a point not in C. Thus, every path connected set is contained in exactly one path component.

Examples.   • A path connected set has only one path component.

• $\mathbb{R} \smallsetminus \{0\}$ has exactly two path components: $(-\infty, 0)$ and $(0, \infty)$.

• The topologist's sine curve has exactly two path components: the graph of $\sin(1/x)$ and the vertical line segment $\{0\} \times [0, 1]$.

We have seen that path components are the maximal path connected subsets of a space. We may also consider maximal connected subsets of a space.

Definition 6. Let $a, b \in X$. We say a is *connected to* b if there exists a connected subset $C \subset X$ containing both a and b.

Example. If $\phi \colon I \to X$ is a path from a to b, then $\phi(I)$ is a connected subset containing a, b. Hence, a is connected to b.

Theorem 7. Suppose X is a space such that any two points are connected. Then X is connected.

Proof. Let $f \colon X \to \mathbb{R}$ be continuous, and suppose there exist $a, b \in X$ such that $f(a) < 0$ and $f(b) > 0$. Then there exists $C \subset X$, a connected subspace, such that $a, b \in C$. Since C is connected, there exists $b \in C$ such that $f(b) = 0$.   □

Proposition 8. Let X be a topological space. Define $a \sim b$ if a is connected to b. Then $\sim$ is an equivalence relation on X.

Proof. Reflexive: The subspace $\{a\} \subset X$ is connected, hence $a \sim a$.
    Symmetric: $a \sim b \iff a, b \in C$ for some connected set $C \iff b \sim a$.
    Transitive: Suppose $a \sim b$ and $b \sim c$. Then there exist connected sets Y, Z such that $a, b \in Y$ and $b, c \in Z$. Suppose that $f \colon Y \cup Z \to \mathbb{R} \smallsetminus \{0\}$ is continuous. If $f(b) > 0$, then $f(Y) \subset (0, \infty)$ and $f(Z) \subset (0, \infty)$, hence $f(Y \cup Z) \subset (0, \infty)$. Similarly, if $f(b) < 0$, then $f(Y \cup Z)$ has only negative values. Hence, $Y \cup Z$ is connected. Since $a, c \in Y \cup Z$, $a \sim c$.   □

Definition 9. The equivalence classes of this relation are called the *connected components*, or simply *components*, of the space X.

By definition, a is connected to b iff a and b lie in the same component. X is the disjoint union of its components. If C is a component, then for all $a, b \in C$, a is connected to b; by Theorem 7, C is connected.
    If $C \subset X$ is connected, then any two points of C are connected; hence, C is contained in a (unique) component. In particular, every path component of X is contained in a unique component.

4

If $X = U \cup V$ is a separation of X, and $C \subset X$ intersects both U and V, then $U \cap C$, $V \cap C$ form a separation of C. Thus, every connected subset, and in particular every component of X, is contained in either U or V.

Examples.
- A set is connected iff it has exactly one component. In particular, the topologist's sine curve has exactly one component.

- $\mathbb{R} \smallsetminus \{0\}$ has $(-\infty, 0)$ and $(0, \infty)$ as its components.

- Let $A \subset \mathbb{Q}$, with the topology induced as a subspace of $\mathbb{R}$, and suppose that A has at least two distinct points a, b. Since the irrationals are dense in $\mathbb{R}$ (e.g., irrationals of the form $q\sqrt{2}$, where $q \in \mathbb{Q}$, are dense), there exists an irrational $c \in (a, b)$. Hence, $((-\infty, c) \cap A)$ and $((c, \infty) \cap A)$ form a separation of A. Thus, the only connected subsets of $\mathbb{Q}$ are the points, and so the points are the components.

- By similar reasoning, $\mathbb{R} \smallsetminus \mathbb{Q}$ has its components all singletons; consequently, it has uncountably many components.

- Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $K \cup \{0\}$ has infinitely many components.

4 Useful theorems

Finally, we present four theorems (or lemmas, if you prefer) that can be helpful in proving sets are connected. The proofs of a couple of these are easier using components, although certainly they can be proved without using components (and are in Munkres).

Theorem 10. A finite product of connected spaces is connected.

Proof. By induction, it suffices to show that if X, Y are connected, so is $X \times Y$. Let $(a, b), (c, d) \in X \times Y$. The function

$$f \colon X \to X \times Y$$
$$x \mapsto (x, b)$$

is a homeomorphism onto its image, so $f(X) = X \times \{b\}$ is connected. Since $(a, b), (c, b)$ both lie in the connected set $X \times \{b\}$, the two points are connected. Similarly, (c, b) is connected to (c, d). Since any two points are connected, $X \times Y$ has only one component, i.e., is connected. $\square$

Theorem 11. If a collection of connected subspaces of X have a point in common, then their union is connected.

Proof. Let $\{A_\alpha\}$ be such a collection, with $x \in \bigcap_\alpha A_\alpha$. Then whenever $y \in A_\alpha$ for any α, the points y and x are connected by A_α inside $\bigcup_\alpha A_\alpha$. Thus, for every $y \in \bigcup_\alpha A_\alpha$, y lies in the same component as x, and so $\bigcup_\alpha X_\alpha$ has only one component. $\square$

Theorem 12. The continuous image of a connected space is connected.

Proof. Let $f\colon X \to Y$ be a continuous, and suppose $f(X)$ is disconnected. Then there exists a continuous map $g\colon f(X) \to \mathbb{R} \smallsetminus \{0\}$ taking both positive and negative values. Hence $g \circ f\colon X \to \mathbb{R} \smallsetminus \{0\}$ is a continuous map taking both positive and negative values, and so X is disconnected. $\square$

Theorem 13. If $A \subset X$ is connected, so is $\bar{A}$.

Proof. [Note: we proved this last lecture, but I am including it in the handout here for completeness.] Let $f\colon \bar{A} \to \mathbb{R} \smallsetminus \{0\}$ be continuous. Since A is connected, $f(A) \subset (-\infty, 0)$ or $f(A) \subset (0, \infty)$. Thus, $f(\bar{A}) \subset \overline{f(A)}$, which is contained in either $(-\infty, 0)$ or $(0, \infty)$ since these sets are closed in $\mathbb{R} \smallsetminus \{0\}$. $\square$