Theorem 1. Let X be a topological space. The following are equivalent:

(i) X is connected.

(ii) If $f\colon X \to \mathbb{R}$ is continuous and there exist $a, b \in X$ such that $f(a) < 0$ and $f(b) > 0$, then there exists $c \in X$ such that $f(c) = 0$.

(iii) No continuous map $f\colon X \to \mathbb{R} \setminus \{0\}$ takes both positive and negative values.

Proof. (ii) says "A continuous function $X \to \mathbb{R}$ that takes both positive and negative values has a zero." (iii) says "A continuous function $X \to \mathbb{R}$ that has no zero does not take both positive and negative values." These are contrapositives, so $(ii) \iff (iii)$.

$(i) \implies (iii)$: Suppose $f\colon X \to \mathbb{R}\setminus\{0\}$ is continuous. Then $U = f^{-1}((0,\infty))$, $V - f^{-1}((-\infty, 0))$ are open sets, and

$$U \cup V = f^{-1}((0,\infty) \cup (-\infty, 0)) = X$$
$$U \cap V = f^{-1}((0,\infty) \cap (-\infty, 0)) = \emptyset.$$

Since X is connected, this implies U or V must be empty.

$(iii) \implies (i)$: Assume X is disconnected. Let $X = U \cup V$ be a separation of X. Define $f\colon X \to \mathbb{R} \setminus \{0\}$ by

$$f(x) = \begin{cases} -1 & \text{if } x \in U, \\ 1 & \text{if } x \in V. \end{cases}$$

By the local criterion for continuity, f is continuous. Since U, V are both non-empty, f takes positive and negative values. $\square$

Lemma 2. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous map such that $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in [a,b]$ such that $f(b) = 0$. (i.e., $\mathbb{R} \setminus \{0\}$ is not path connected.)

Proof. Let $A = f^{-1}((-\infty, 0))$, $B = f^{-1}([0,\infty))$. Thus, $\overline{f(A)} \subset (-\infty, 0]$, $\overline{f(B)} \subset [0,\infty)$. Let $c = \sup A$. Thus, $c \in \overline{A}$. Since $f(\overline{A}) \subset \overline{f(A)}$, we see that $f(c) \leq 0$.

Since $f(b) > 0$, we know $c \neq b$. Let $(d,e) \subset [a,b]$ be an open interval about c, and let $x \in (d,e)$. Since c is an upper bound on A, $x \notin A$; hence, $x \in B$. Since an arbitrary open interval about c intersects B, $c \in \overline{B}$. Hence, $f(c) \in \overline{f(B)} \subset [0,\infty)$, i.e., $f(c) \geq 0$. Therefore, $f(c) = 0$. $\square$

Theorem 3. Any path connected space is connected.

Proof. Let X be path connected, $a, b \in X$, and $f\colon X \to \mathbb{R}$ a continuous map such that $f(a) < 0$, $f(b) > 0$. Let $\phi\colon I \to X$ be a path from a to b. Then $f \circ \phi\colon [0,1] \to \mathbb{R}$ is a continuous map, $(f \circ \phi)(0) < 0$, and $(f \circ \phi)(1) > 0$. By the Lemma, there exists $c \in [0,1]$ such that $(f \circ \phi)(c) = 0$. Then $\phi(c) \in X$ is a zero of f. $\square$