Math 133, Lecture 1: Introduction to Differential Equations

Charles Staats

Monday, 26 March 2012

Some logistics

- The first tutorial will be on Thursday, March 29. You will be assigned to tutorial sessions by then. Please email me if you have a preference between the two tutors. I will try to accommodate requests, but I can make no guarantees.
- I will be taking additional steps to encourage tutorial attendance this quarter. These will include the following:
  - Pop quizzes will be given at the beginning of some tutorials. These quizzes are intended to be easy for anyone who was paying attention in class. They will count toward your final average.
  - Tutors will take attendance and report it to me. I will contact anyone who is frequently absent.
  - Certain "hard" problems in the homework will be designed to be discussed in tutorial.

  If you feel that tutorial is a waste of your time, please talk to your tutor or to me about it. (If you talk to me, I will not share your name with your tutor.)
- I'm going to wait to select office hours until I have a better idea of my own schedule. I will set them on Saturday at the latest. In the mean time, feel free to email me and ask for appointments; I'm always happy to help.
- I will specify, on homework sets, that certain problems "will be graded carefully." At the discretion of your tutor, the remaining problems may be graded for completion only. However, you should still hand them in unless the homework set specifically says not to.
- The first assignment will be due Friday. I have not yet selected it, but I soon will.

1 What is a differential equation?

In algebra, you take equations relating x and y, and try to put one variable in terms of the other. For instance, you might take the equation

$$x^2 + y^2 = 1$$

and solve for y in terms of x, obtaining

$$y = \pm\sqrt{1 - x^2}.$$

There are plenty of equations that we do not have the means to solve—or at least, the solution cannot be expressed as a formula. For instance, try solving

$$y + \sin y = x$$

for y. There is a "solution" in the sense that y can be given as a *function* of x; however, there is no *formula* for this function. This, as you may recall, was one of my primary motivations for introducing functions way back at the beginning of first quarter: sometimes, we have solutions that cannot be given by a formula, so we talk about functions rather than formulas.

In calculus, we can talk about solving a whole new kind of equation: *differential* equations.

Definition. A *differential equation* is an equation relating x, y, and dy/dx. (Note that the term also applies if the two variables are called something other than x and y.)

[Note that I am not telling you the entire truth here. Technically, a differential equation can also involve higher derivatives—$d^2y/dx^2, d^3y/dx^3, \ldots$. It can even involve more than two variables. However, unless something comes up that I do not expect, the differential equations we deal with will all fit into the definition above.]

Solving a differential equation is a matter of finding all functions f such that the equation is satisfied when $y = f(x)$.

Example 1. Solve the differential equation

$$\frac{dy}{dx} = x.$$

Solution. $y = \frac{1}{2}x^2 + C$, for any real number C. □

The simplest type of differential equation, as demonstrated above, is of the form

$$\frac{dy}{dx} = [\text{function in } x].$$

Solving these is simply a matter of finding antiderivatives, i.e., indefinite integrals. However, even for these "simple" equations, there is not always a solution given by a formula.

Example 2. The differential equation

$$\frac{dy}{dx} = e^{-x^2}$$

has a solution (as a function), which is extremely important in statistics and probability. However, this function cannot be given by a formula.

2 Separation of Variables

The examples of differential equations given in the previous section are really just slightly different ways of restating the problem of "indefinite integration." However, there are plenty of important differential equations that are not quite so simple.

Example 3. Solve the differential equation

$$\frac{dy}{dx} = y.$$

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= y \\
\frac{1}{y}\,dy &= dx \\
\int \frac{1}{y}\,dy &= \int dx \\
\ln|y| + C_1 &= x + C_2 \\
\ln|y| &= x + C_2 - C_1 = x + C_3 && \text{where } C_3 := C_2 - C_1 \\
|y| &= e^{x+C_3} = e^{C_3}e^x \\
y &= \pm e^{C_3}e^x = Ce^x,
\end{aligned}$$

where $C := \pm e^{C_3}$. □

Note that in solving equations like this one, we freely re-assign the names of our constants to make the formulas look nicer. Also note that we have to be more careful with the constants than we were last quarter; adding "$+C$" once you've done everything else will not work.

This particular technique for solving differential equations is called *separation of variables*. Here's the idea: if we can somehow rewrite our differential equation in the form

$$f(y)\,dy = g(x)\,dx,$$

we have "separated" the variables and can integrate the two sides separately. Like all techniques for solving differential equations, this one does not always work. But when it does work, it can be extremely useful.

Example 4. Solve the differential equation

$$\frac{dy}{dx} = xy.$$

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= xy \\
\frac{1}{y}\,dy &= x\,dx \\
\int \frac{1}{y}\,dy &= \int x\,dx \\
\ln|y| + C_1 &= \tfrac{1}{2}x^2 + C_2 \\
\ln|y| &= \tfrac{1}{2}x^2 + C_3 \\
|y| &= e^{C_3} e^{\frac{1}{2}x^2} \\
y &= Ce^{\frac{1}{2}x^2}.
\end{aligned}$$

□

3 Differential equations in real life

The reason differential equations are so important is that they come up often in problems about the "real world"—in physics, chemistry, and engineering, but also in population studies, economics, medicine, and any number of other things. (I once used a differential equation to help me understand a computer game.) I will be putting more emphasis than the textbook on how to translate a real-world problem into a differential equation, and less emphasis on actually solving the differential equation (in part because so many differential equations cannot be solved by formulas). Consequently, I will be writing significant portions of the first two assignments myself, which is part of why I have not yet selected them.

Example 5. Assume that in a given population (of bacteria, or of people), the population P grows according to the following law:

> The rate of growth is directly proportional to the number of people already present.

State this law as a differential equation.

Solution. The rate of growth of population (with respect to time) is dP/dt. The "number of people already present" is the population, P. So, the law states that dP/dt is directly proportional to P. As an equation,

$$\frac{dP}{dt} = kP,$$

where k is the *constant of proportionality*. Note that the information we have been given is not enough to determine k. $\square$

Example 6. ("density/collision problem") A disease is spreading through a population of $10,000$ people. Let I be the number of people infected. Any time an infected person meets an uninfected person, there is a chance that the uninfected person becomes infected.

If we assume that these meetings are completely "at random," then the frequency of such meetings should be

- directly proportional to the number I of infected people, and
- directly proportional to the number $10,000 - I$ of uninfected people.

Consequently, it should be directly proportional to their product, and we have the differential equation

$$\frac{dI}{dt} = k \cdot I \cdot (10,000 - I).$$

Again, k is a constant of proportionality that we cannot determine without further data.

Although the assumption that "meetings are completely random" is clearly false, disease epidemics are, surprisingly enough, often described by the sort of differential equation given in the example above. This sort of equation is called an equation for *logistic growth*; we may have time to discuss it more later.

Math 133, Lecture 2: What to do with a DifEq once you've solved it

Charles Staats

Wednesday, 28 March 2012

1 How to use differential equations to get actual information

Suppose you are given a real life situation and asked to write a differential equation. For instance, someone might tell you that a particular population grows at a rate proportional to the existing population. The corresponding differential equation is

$$\frac{dP}{dt} = kP,$$

where k is the constant of proportionality, t represents time, and $P(t)$ is the population at time t. If we solve this differential equation, we end up with

$$P(t) = Ce^{kt}.$$

Unfortunately, this does not actually tell us what the population is at any given time: we still have two constants, C and k, floating around, and no one has told us what their values are. This is actually a good thing: the differential equation $dP/dt = kP$ can describe all sorts of populations, from bacteria in a Petri dish to world population. If this equation, by itself, were enough to completely determine the function P, then all of these different populations would be determined by it, and hence be equal to each other—which is absurd. However, it does leave us with something of a conundrum: how can we actually calculate population at a given time? The answer is that we need more data. Typically, for each constant that shows up in the equation, we need one data point (t_0, P_0) telling us what the population is at some particular time t_0.

Here's the general procedure for how to proceed when you are given a real-world problem relating x and y, including both "rate of change" information and one or more data points (x_0, y_0).

1. Write a differential equation relating x, y, and dy/dx.

2. Solve it. The resulting "solution" will express y as a function of x, but will probably include at least one constant you don't know the value of.

3. Substitute in known values of x and y and solve for the constant(s).

Example 1. The number of bacteria P in a Petri dish started out at $5,000$, but was equal to $10,000$ after 3 hours. Assuming that the population growth rate is proportional to the existing population, find P as a function of time t.

Solution. We are told that "the population growth rate is proportional to the existing population;" we can translate this into the differential equation

$$\frac{dP}{dt} = kP,$$

for some constant of proportionality k. We are also given the data points

$$\begin{aligned} P(0) &= 5000 \\ P(3) &= 10000. \end{aligned}$$

Let's follow the procedure.

1. **Write a differential equation relating x, y, and dy/dx.** We just did this:
$$\frac{dP}{dt} = kP.$$

2. **Solve it.** We use separation of variables:
$$\begin{aligned} \frac{dP}{dt} &= kP \\ \frac{dP}{P} &= k\ dt \\ \int \frac{dP}{P} &= \int k\ dt \\ \ln|P| + C_1 &= kt + C_2 \\ \ln|P| &= kt + C_3 \\ |P| &= e^{kt+C_3} = e^{C_3}e^{kt} \\ P &= \pm e^{C_3}e^{kt} = Ce^{kt}. \end{aligned}$$

   We find that $P(t) = Ce^{kt}$, for some constants C and k.

3. **Substitute in known values of x and y and solve for the constant(s).** Our two data points give
$$\begin{aligned} P(0) &= 5000 & \qquad P(3) &= 10000 \\ Ce^{k\cdot 0} &= 5000 & \qquad Ce^{3k} &= 10000. \end{aligned}$$

   Since $e^{k\cdot 0} = e^0 = 1$, the first equation gives
$$5000 = Ce^{k\cdot 0} = C \cdot 1 = C.$$

Substituting $C = 5000$ into the second equation gives

$$\begin{aligned} 5000e^{3k} &= 10000 \\ e^{3k} &= 10000/5000 = 2 \\ 3k &= \ln 2 \\ k &= \tfrac{1}{3}\ln 2. \end{aligned}$$

Thus, we have

$$P(t) = Ce^{kt} = 5000e^{\frac{1}{3}t\ln 2}. \qquad \square$$

Assignment 1: due Friday, March 30

The problems from the handout. They will all be graded carefully. Hopefully, they will go reasonably quickly, once you get the hang of them.

Assignment 2: due Monday, April 2

Section 3.9, Problems 1, 2, 7, and 8. Problems 2 and 8 will be graded carefully.

Section 6.5, Problems 1, 2, 5, and 6. Problems 2 and 6 will be graded carefully.

Math 133, Lecture 3: Linear differential equations; Inverse Trig Functions

Charles Staats

Friday, 30 March 2012

1 Linear Differential Equations

Recall that a differential equation is *separable* if we can write it as

$$f(y)\,dy = g(x)\,dx.$$

If a differential equation is separable, we can solve it (or attempt to solve it) by writing it in the form above and taking indefinite integrals of both sides.

In general, "solution techniques" to differential equations involve the following idea: if we can put a differential equation in a certain form, then we can apply a certain trick that lets us solve it just by taking antiderivatives (i.e., indefinite integrals). Separability is one of the simplest of these: if we can put the differential equation into the form $f(y)\,dy = g(x)\,dx$, then we can solve it by integrating both sides.

Another form of equation that has a "trick" for solving it is called a *linear differential equation.*

Definition. A *first-order*[1] *linear differential equation* is a differential equation that can be put in the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

for some functions p and q.

Example 1. When you solved the mortgage problem on the homework set, you should have gotten the differential equation

$$\frac{dy}{dx} = .05y - 6000,$$

where y represents the debt and x represents time in years. If we rewrite this as

$$\frac{dy}{dx} + (-.05)y = -6000,$$

[1] "First-order" just means that first derivatives appear, but no second derivatives, third derivatives, etc.

we have put it into the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

where $p(x) = -.05$ and $q(x) = -6000$ are both constant functions. Thus, we see that this differential equation is a first-order linear differential equation.

The technique (or "trick") for solving differential equations of this form relies on the following:

Exercise 2. Let $p(x)$ be a function of x, and suppose that P is an antiderivative of p. In other words,

$$\begin{aligned} P'(x) &= p(x) \\ \int p(x)\ dx &= P(x) + C. \end{aligned}$$

Show that

$$\frac{d}{dx}\left(y \cdot e^{P(x)}\right) = e^{P(x)}\left(\frac{dy}{dx} + p(x)y\right).$$

Solution:

$$\begin{aligned} \frac{d}{dx}\left(ye^{P(x)}\right) &= \frac{dy}{dx}e^{P(x)} + y\frac{d}{dx}e^{P(x)} \\ &= \frac{dy}{dx}e^{P(x)} + ye^{P(x)}P'(x) \\ &= \frac{dy}{dx}e^{P(x)} + ye^{P(x)}p(x) \\ &= e^{P(x)}\left(\frac{dy}{dx} + p(x)y\right). \end{aligned}$$

□

Now, here's the general technique for solving a differential equation of the form $y' + p(x)y = q(x)$:

1. Find an antiderivative $P(x) = \int p(x)\ dx$.
2. Multiply both sides of the differential equation by $e^{P(x)}$. Then we get

$$\begin{aligned} \frac{dy}{dx} + p(x)y &= q(x) \\ e^{P(x)}\left(\frac{dy}{dx} + p(x)y\right) &= e^{P(x)}q(x). \end{aligned}$$

By the exercise,

$$
\begin{aligned}
\frac{d}{dx}\left(y \cdot e^{P(x)}\right) &= e^{P(x)} q(x) \\
y \cdot e^{P(x)} &= \int e^{P(x)} q(x)\ dx \\
y &= e^{-P(x)} \int e^{P(x)} q(x)\ dx.
\end{aligned}
$$

The last step is obtained by multiplying both sides by $e^{-P(x)}$, which is equivalent (by properties of exponents) to dividing by $e^{P(x)}$.

Example 3. Consider the differential equation

$$\frac{dy}{dx} + (-.05)y = -6000$$

that came from the mortgage problem. Here we have

$$
\begin{aligned}
p(x) &= -.05 \\
\int p(x)\ dx &= -.05x + C,
\end{aligned}
$$

so $P(x) = -.05x$ is an antiderivative of $p(x)$. Thus, our method tells us that we should multiply both sides of the differential equation by $e^{-.05x}$.

$$
\begin{aligned}
e^{-.05x}\frac{dy}{dx} - .05e^{-.05x}y &= -6000e^{-.05x} \\
\frac{d}{dx}\left(e^{-.05x}y\right) &= -6000e^{-.05x} \\
e^{-.05x}y &= -6000 \int e^{-.05x}\ dx.
\end{aligned}
$$

Let

$$
\begin{aligned}
u &= -.05x \\
du &= -.05\ dx \\
-20u &= dx.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\int e^{-.05x}\ dx &= \int e^u \cdot (-20)\ du \\
&= -20 \int e^u\ du = -20e^u + C_1 = -20e^{-.05x} + C_1.
\end{aligned}
$$

Consequently,

$$\begin{aligned} e^{-.05x}y &= -6000 \int e^{-.05x}\,dx \\ &= -6000(-20e^{-.05x} + C_1) \\ &= 120,000e^{-.05x} + C_2 \\ y &= e^{.05x}(120,000e^{-.05x} + C_2) \\ &= 120,000 + C_2e^{.05x}. \end{aligned}$$

We end up with

$$y = 120,000 + Ce^{.05x}$$

for some constant C.

Exercise 4. In the original mortgage problem, the debt starts out at \$100,000; in other words, at time zero (when $x = 0$), we should have $y = 100,000$. Use this data point to find the value of the constant C and express y as a function of x. Then, use that function to calculate how long (approximately) it will take before the mortgage is paid off.

2 Inverse Trig Functions

Much of mathematics is about solving "inverse problems." For instance: if $y = x^2$, and someone gives us x, we can find y. But we can also "invert" this question: if someone gives us y, how do we find x? In this particular case, we would like to define a new function $\sqrt{\ }$, by

$$\sqrt{x} := \text{"the number whose square is } x\text{."}$$

There are two issues with this definition. The first is that if $x < 0$, then there *is* no number whose square is x. To handle this, we say that the $\sqrt{\ }$ function is only defined on the interval $[0, \infty)$, i.e., on numbers $x \geq 0$.

The second issue is that if $x > 0$, then the rule is ambiguous: there are two different numbers whose square is x. For instance, if $x = 1$, then "the number whose square is 1" could refer to either 1 or -1. Functions, by definition, are not allowed to be ambiguous. To handle this, we only consider *nonnegative* numbers whose square is x—we discard the negative solution. Thus, we obtain the following definition of $\sqrt{\ }$:

Definition. For x in the interval $[0, \infty)$, we define $\sqrt{x}$ to be the number in the interval $[0, \infty)$ whose square is x.

A similar procedure will allow us to define an "inverse" for the trigonometric functions sin, cos, and tan. We denote these "inverses" by arcsin, arccos, and arctan. When we measure angle in radians, the measure of an angle corresponds to the length of the corresponding *arc* on the unit circle. So, you can think of, e.g., $\arcsin y$ as an abbreviated form of "the angle *(arc)* whose *sine* is y." Unfortunately, if we try to define the arcsin function by

$$\arcsin y := \text{"the angle (in radians) whose sine is } y\text{"},$$

we encounter the same two difficulties as we did for defining $\sqrt{x}$. The first difficulty is that the sine of an angle is always in the interval $[-1, 1]$. (If you recall, the sine of the angle is always a y-coordinate on the unit circle.) Consequently, we need to say that arcsin is only defined on the interval $[-1, 1]$.

The second difficulty is that the definition is ambiguous: for any given y-coordinate in $[-1, 1]$, there are infinitely many different angles with this same y. There are two types of ambiguities we need to avoid.

- If we start at angle t, and go all the way around the unit circle (i.e., add 2π to our angle), we obtain another angle $t+2\pi$ that has exactly the same (x, y)–coordinates on the unit circle. In other words, $t + 2\pi$ has exactly the same cosine and sine as t. To avoid this ambituity, we restrict our definition to be "the angle in the interval $(-\pi, \pi]$ whose sine is y." This way, once we specify a point on the unit circle, we have a unique angle it produces.

- Even with this restriction, there can be more than one point on the unit circle with the same sine (i.e., same y-coordinate); see the picture I did not have time to include in these notes, but will draw on the board (and which you should copy down). To avoid this sort of ambitugity, we further restrict our definition to give only angles in $[-\pi/2, \pi/2]$.

Thus, arcsin is defined by

Definition. For y in the interval $[-1, 1]$, we define $\arcsin y$ to be the number in the interval $[-\pi/2, \pi/2]$ whose sine is y.

Assignment 2: due Monday, April 2

Section 3.9, Problems 1, 2, 7, and 8. Problems 2 and 8 will be graded carefully.

Section 6.5, Problems 1, 2, 5, and 6. Problems 2 and 6 will be graded carefully.

Assignment 3: due Wednesday, April 4

Section 6.6, Problems 1, 2, 7, and 8. Problems 2 and 8 will be graded carefully.

Section 6.8, Problems 1 and 2. Problem 2 will be graded carefully.

Exercise 4 in Lecture 3 (p. 4). This problem will be discussed in tutorial on Tuesday. It will be graded carefully.

Math 133, Lecture 4: Inverse Trig Functions, continued

Charles Staats

Monday, 2 April 2012

1 The f^{-1} notation

If f is a function, the "inverse function of f" describes the "solution" to the equation $y = f(x)$. More precisely, the inverse function, usually called f^{-1}, is defined by

$$f^{-1}(y) = \text{the number } x \text{ such that } f(x) = y.$$

Note that this definition does not always make sense: for instance, if $f(x) = x^2$, then $f^{-1}(y)$ is undefined for $y < 0$ (there is no number whose square is y) and ambiguous for $y > 0$ (if $f(x) = y$, then $f(-x) = y$ also, so it is not clear whether $f^{-1}(y)$ should be x or $-x$). There are ways of handling this: we can restrict the domain of f—artificially declare that $f(x)$ is only defined for $x \geq 0$. Then f^{-1} does exist, defined by $f^{-1}(y) = \sqrt{y}$. We have eliminated the ambiguity by throwing out the x-values that give extra solutions for f.

The essential defining property for f^{-1} may be restated as

$$f^{-1}(y) = x \quad \text{if and only if} \quad y = f(x).$$

In other words, if we are trying to do algebra, and we have an unwanted f around one side of the equation, we can get rid of it by applying f^{-1} to both sides of the equation. Likewise, if we have an unwanted f^{-1}, we can get rid of it by applying f to both sides of the equation. Or in other words,

$$\begin{aligned} f^{-1}(f(x)) &= x && \text{whenever } f(x) \text{ is defined, and} \\ f(f^{-1}(y)) &= y && \text{whenever } f^{-1}(y) \text{ is defined.} \end{aligned}$$

If we rewrite these using the language of "composition of functions," we get that

$$\begin{aligned} (f^{-1} \circ f)(x) &= x \\ (f \circ f^{-1})(y) &= y. \end{aligned}$$

Since this is kind of like the statement that $x \cdot x^{-1} = 1 = x^{-1} \cdot x$, where $x^{-1} = 1/x$, it provides some explanation for the notation f^{-1}. Unfortunately, this can produce some confusion.

Warning. *$f^{-1}(x)$ is almost never equal to $(f(x))^{-1}$. The former means "the number y such that $f(y) = x$;" the latter means $1/f(x)$.*

In particular, the inverse trigonometric functions arcsin, arccos, and arctan are often denoted $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, respectively.

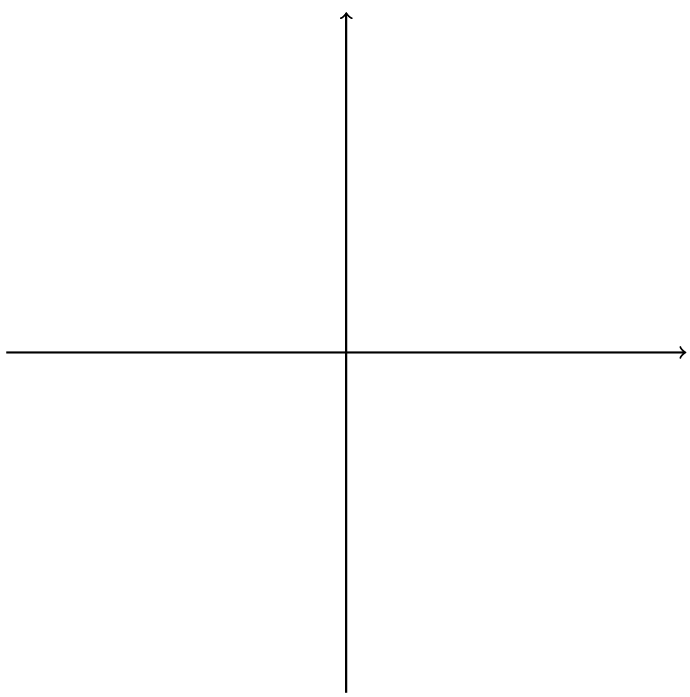
Warning. *While conventional notation allows us to write*

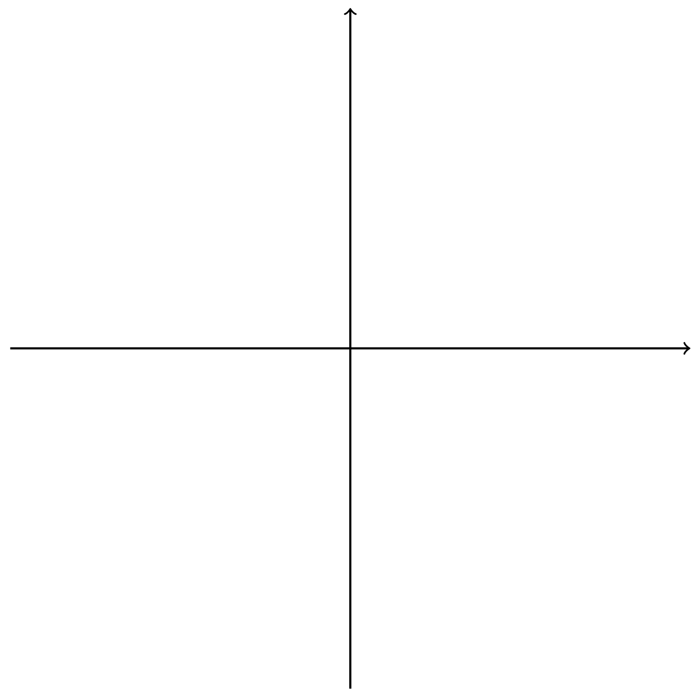
$$\sin^2 x = (\sin x)^2 \qquad \cos^2 x = (\cos x)^2 \qquad \tan^2 x = (\tan x)^2,$$

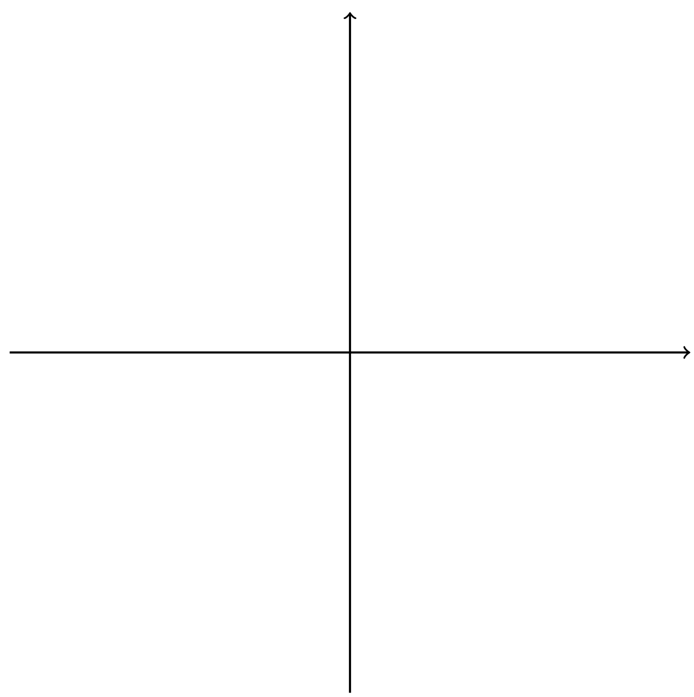
it is most emphatically not true *that*

$$\cancel{\sin^{-1} x = \frac{1}{\sin x}} \qquad \cancel{\cos^{-1} x = \frac{1}{\cos x}} \qquad \cancel{\tan^{-1} x = \frac{1}{\tan x}}$$

2 The graphs of arcsin, arccos, and arctan







3 Identities with inverse trig functions

4 The derivatives of arcsin, arccos, and arctan

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}.$$

Assignment 3: due Wednesday, April 4

Section 6.6, Problems 1, 2, 7, and 8. Problems 2 and 8 will be graded carefully.

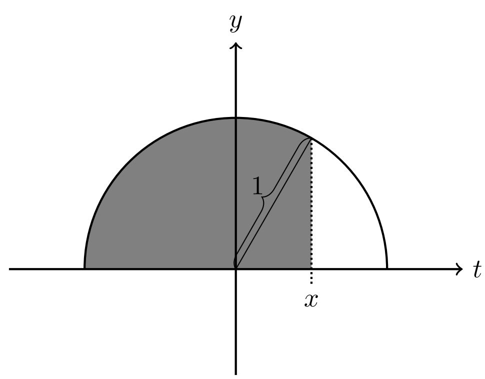
Section 6.8, Problems 1 and 2. Problem 2 will be graded carefully.

Exercise 4 in Lecture 3 (p. 4). This problem will be discussed in tutorial on Tuesday. It will be graded carefully.

Assignment 4: due Friday, April 6

Section 6.8, Problems 29–32 and 39–40. The even-numbered problems will be graded carefully.

Without using calculus, find an expression for the area of the shaded region below:



Then, express the area as a definite integral, without any trigonometric functions (inverse or otherwise). You do not need to evaluate the definite integral—you have already done that, without needing to use calculus. This problem will be discussed in tutorial on Thursday. It will be graded carefully.

Math 133, Lecture 5: Derivatives of the inverse trig functions; Integration via trig substitution

Charles Staats

Wednesday, 4 April 2012

1 Office Hours

My weekly office hours will be

- Monday, 3:15 to 4:15
- Thursday, 1:45 to 2:45
- Friday, 4:15 to 5:15.

I am also open to appointments.

2 Derivatives of the inverse trig functions

First, let's have a few exercises to get warmed up.

Exercise 1. Find an expression for $\cos(\arcsin x)$ that does not involve trigonometric functions.

Solution.

$$\frac{1}{\sqrt{1-x^2}}$$

□

Exercise 2. Find an expression for $\sin(\arccos x)$ that does not involve trigonometric functions.

Solution.

$$\frac{1}{\sqrt{1-x^2}}$$

□

Exercise 3. Find an expression for $\cos(\arctan x)$ that does not involve trigonometric functions.

Solution.
$$\frac{x}{\sqrt{1+x^2}}$$
$\square$

Now, let's find the derivatives of arcsin, arccos, and arctan, via the technique of "implicit differentiation."

Example 4. Find $\frac{d}{dx} \arcsin x$.

Solution.

$$\begin{aligned} y &= \arcsin x \\ \sin y &= x. \end{aligned}$$

Differentiating both sides with respect to x,

$$\begin{aligned} \frac{d}{dx} \sin y &= \frac{d}{dx} x \\ \frac{d}{dy}(\sin y) \cdot \frac{dy}{dx} &= 1 \\ \cos y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\cos(\arcsin x)} \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$
$\square$

Exercise 5. Show that

$$\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}}.$$

Solution.

$\square$

NOTE: Recall that

$$\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}.$$

Example 6. Find $\dfrac{d}{dx}\arctan x$.

Solution.

$$\begin{aligned}
y &= \arctan x \\
\tan y &= x \\
\frac{d}{dx}\tan y &= \frac{d}{dx}x \\
\frac{1}{\cos^2 y}\cdot\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \cos^2 y \\
&= (\cos y)^2 \\
&= (\cos\arctan x)^2 \\
&= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 \\
&= \frac{1}{1+x^2}.
\end{aligned}$$

□

3 Corresponding integral rules

We've just found several "new" differentiation rules. This allows us to write down corresponding rules for finding antiderivatives, i.e., definite integrals.

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C \qquad (1)$$

$$\int \frac{du}{1+u^2} = \arctan u + C. \qquad (2)$$

Together with the rule

$$\int \frac{du}{u} = \ln|u| + C, \qquad (3)$$

these might be termed the "surprising rules" of integration: you start with a perfectly nice, simple function like $1/(1+x^2)$ or $1/x$, and when you go to integrate it, a "transcendental" function like arctan or ln pops out of nowhere.

Now, mathematicians have a sort of a love-hate relationship with surprises. On the one hand, having an unexpected solution pop out of our logical, systematic reasoning is exciting, even the sort of thing that many of us live for. On the other hand, we can't help but feel that if we really, truly understood what was going on, this "surprising" result would not be so surprising. Thus, surprises are an opportunity to delve deeper into the subject: to understand why we should have expected this surprising result all along, if only we were wiser. Part of the difficulty with explaining the joy of mathematics to non-mathematicians is that few except mathematicians ever have the joy of making this sort of discovery for themselves.

In the case of $\int u^{-1}\, du = \ln|u| + C$, there's really not much to be done, as far as I know. The power rule for integration in this case would give us $\frac{1}{0}u^0$, which makes no sense, so there is no option but to invent a new function, called ln, and define it to be the integral of u^{-1}. And if you recall, this is how ln was defined to begin with.

On the other hand, for the "inverse trig" rules, there is a more general technique, called *trigonometric substitution*. Like the best discoveries, this one not only "explains" the previously surprising results, but allows us to extend them to integrate other kinds of functions as well.

4 Trig substitution

The key to trig substitution is the Pythagorean identity

$$\cos^2 x + \sin^2 x = 1;$$

dividing both sides by $\cos^2 x$ gives the sister identity

$$1 + \tan^2 x = \sec^2 x,$$

which is at least as important for our purposes—perhaps even more so.

The key idea is that we can use these identities, together with u-substitution, to get rid of the expressions $1 + x^2$, $1 - x^2$, and $x^2 - 1$ when they show up in unfortunate places—say, under a $\sqrt{}$ symbol, or in the denominator (or both). Applying a trigonometric u-substitution can allow us to condense one of these expressions into a single term via the Pythagorean identities. Here's the basic technique:

- To eliminate $1 + x^2$, substitute $x = \tan u$. Then

$$1 + x^2 = 1 + \tan^2 u = \sec^2 u.$$

- To eliminate $1 - x^2$, substitute $x = \sin u$. Then

$$1 - x^2 = 1 - \sin^2 u = \cos^2 u.$$

- To eliminate $x^2 - 1$, substitutie $x = \sec u$. Then

$$x^2 - 1 = \sec^2 u - 1 = \tan^2 u.$$

Here's an example of an integral for which these techniques are applicable, that does not appear on our list inspiring the techniques.

Example 7.

$$\int \sqrt{1 - x^2}\, dx.$$

Solution. Substitute

$$\begin{aligned} x &= \sin u \\ dx &= \cos u \, du. \end{aligned}$$

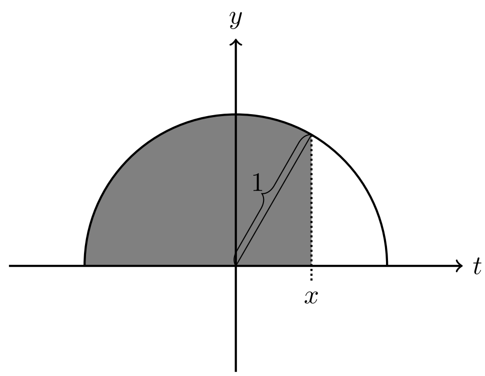
Then we have

$\square$

Assignment 4: due Friday, April 6

Section 6.8, Problems 29–32 and 39–40. The even-numbered problems will be graded carefully.

Without using calculus, find an expression for the area of the shaded region below:



Then, express the area as a definite integral, without any trigonometric functions (inverse or otherwise). You do not need to evaluate the definite integral—you have already done that, without needing to use calculus. This problem will be discussed in tutorial on Thursday. It will be graded carefully.

Assignment 5: due Monday, April 9

Section 6.8, Problems 19, 20, 43, 44, 61, and 62. Problems 20, 44, and 62 will be graded carefully.

Find the derivative of $x \ln x$ with respect to x. Bonus: find an antiderivative of $\ln x$. (Hint: take $x \ln x - g(x)$, for some very simple function g.)

This will be graded carefully.

Math 133, Lecture 6: Integration techniques—review

Charles Staats

Monday, 9 April 2012

1 Basic rules of antidifferentiation

For every basic rule of differentiation, there is a corresponding basic rule of antidifferentiation. The book has what its authors consider a "short list" of such rules (only seventeen items, after all) on pp. 383-84. I agree with the textbook authors that the basic rules should be not only memorized, but ingrained to the point that they can be used without thinking. However, I think their list is a bit long, so I will offer a shorter one.

Exercise 1. (Powers: two cases)

$$\int u^r \, du = \begin{cases} \frac{1}{r+1} u^{r+1} + C & \text{if } r \neq -1, \\ \ln|u| + C & \text{if } r = -1. \end{cases}$$

Exercise 2. (sin and cos)

$$\int \sin u \, du = -\cos u + C$$
$$\int \cos u \, du = \sin u + C.$$

Exercise 3. (exponentials)

$$\int e^u \, du = e^u + C$$
$$\int a^u \, du = \frac{a^u}{\ln a} + C.$$

Note that the second rule only applies for a a positive real number not equal to one. (If a is not positive, a^u is not defined for many u. If $a = 1$, the rule will not make sense.)

2 Techniques of antidifferentiation

For finding derivatives, you had some basic rules to differentiate simple functions like x^n or $\sin x$, together with a few general techniques—most notably, the chain rule and the product rule—stating how, when you see a complicated function that is assembled from simple functions, you can combine the simple rules to obtain the derivative.

For integration, the first concept is the same: every simple rule of differentiation gives you a correspondingly simple rule for antidifferentiation. The second step—applying general techniques to deal with more complicated functions—is also there, but only after a fashion. There are essentially two general techniques for integration: u-substitution (which corresponds to the chain rule) and integration by parts (which you have not yet seen; it corresponds to the product rule for differentiation). Unfortunately, these "general techniques" are not nearly as powerful as the corresponding differentiation techniques. You can use them to change one integral into another, but you have to use them very carefully, guided by experience (or rote memorization of an extensive list of possibilities), in order to make sure that the new integral is actually more tractable than the first. When the general techniques don't work, there is a third alternative: a "bag of tricks" that may allow you to rewrite a function as another, more tractable function.

To recap:

1. Every rule of differentiation gives a corresponding rule of antidifferentiation. For instance, the differentiation rule $D_x \sin x = \cos x$ gives the corresponding integration rule

$$\int \cos x \, dx = \sin x + C.$$

2. If you are asked to do an integral for which none of the basic rules applies, you may be able to use the techniques of u-substitution (which you have seen) and/or integration by parts (which you will see) to transform the integral into a simpler integral. For instance, consider

$$\int \cos 2x \, dx.$$

If we let $u = 2x$, so that $du = 2\ dx$ and $\frac{1}{2}du = dx$, we find that

$$\begin{aligned}\int \cos 2x\ dx &= \int \cos u\ \tfrac{1}{2}du \\ &= \frac{1}{2}\int \cos u\ du\end{aligned}$$

Now, a basic rule applies, telling us that this is

$$\begin{aligned}&= \frac{1}{2}\sin u + C \\ &= \frac{1}{2}\sin 2x + C.\end{aligned}$$

3. Even if there is no obvious way to apply u-substitution or integration by parts, it may be possible to use "tricks" to reveal less obvious ways. For instance, consider the integral

$$\int \cos^2 x\ dx,$$

which we encountered at the end of the last lecture. To integrate this, we apply the trig identity[1]

$$\cos^2 x = \tfrac{1}{2}(\cos 2x + 1).$$

Applying this identity gives us

$$\begin{aligned}\int \cos^2 x\ dx &= \tfrac{1}{2}\int (\cos 2x + 1)\ dx \\ &= \tfrac{1}{4}\int \cos(2x)\cdot(2\ dx) + \tfrac{1}{2}\int 1\ dx \\ &= \tfrac{1}{4}\sin(2x) + \tfrac{1}{2}x + C.\end{aligned}$$

Note that the substitution $u = 2x$, $du = 2\ dx$ was used implicitly in going from the second to the third line.

3 Some examples

I'll give three examples in this section. The first two will be examples from the textbook, intended to refresh you on u-substitution techniques you should already be familiar with—at least in principle. The third example will be more complex, dealing with some unfinished business from last lecture.

[1] You will need to have this identity memorized, or at least be able to derive it, for the test; but you should not feel bad if you do not recognize it—I'm not sure you have ever seen it before.

Example 4. (Example 5, p. 385 in the textbook)

$$\int x\cos x^2\,dx = \frac{1}{2}\int \cos u\,du \qquad u = 2x, \quad du = 2x\,dx, \quad x\,dx = \tfrac{1}{2}\,du$$
$$= \tfrac{1}{2}\sin u + C$$
$$= \tfrac{1}{2}\sin x^2 + C.$$

Example 5. (Example 3, p. 384 in the textbook)

$$\int \frac{6e^{1/x}}{x^2}\,dx.$$

Try getting rid of the thing inside the exponent. Set $u = 1/x = x^{-1}$, so that $du = -x^{-2}dx$. Then we have

$$\begin{aligned}\int \frac{6e^{1/x}}{x^2}\,dx &= \int 6e^{1/x}\cdot x^{-2}\,dx\\ &= \int 6e^u\cdot(-\,du)\\ &= -6\int e^u\,du\\ &= -6e^u + C\\ &= -6e^{1/x} + C.\end{aligned}$$

Example 6. (Demonstration of Trig Substitution, from last time)

$$\int \sqrt{1-x^2}\,dx$$

Solution. Recall that the Pythagorean identity

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= 1\\ \cos^2\theta &= 1-\sin^2\theta\end{aligned}$$

suggests that to simplify an expression like $1-x^2$, we should substitute $x = \sin\theta$. Then we get

$$\begin{aligned}x &= \sin\theta\\ dx &= \cos\theta\,d\theta,\end{aligned}$$

and so

$$\begin{aligned}
\int \sqrt{1-x^2}\; dx &= \int \sqrt{1-\sin^2\theta} \cdot \cos\theta \; d\theta \\
&= \int \sqrt{\cos^2\theta} \cdot \cos\theta \; d\theta \\
&= \int \cos\theta \cdot \cos\theta \; d\theta \\
&= \int \cos^2\theta \; d\theta.
\end{aligned}$$

Recalling the identity $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$, the above is

$$\begin{aligned}
&= \int \tfrac{1}{2}(\cos 2\theta + 1)\; d\theta \\
&= \tfrac{1}{4} \int \cos 2\theta \cdot 2\; d\theta + \tfrac{1}{2} \int 1 \; d\theta.
\end{aligned}$$

substituting $u = 2\theta$ and $du = 2\; d\theta$ in the first integral, we get

$$= \tfrac{1}{4} \sin 2\theta + \tfrac{1}{2}\theta + C.$$

Finally, going back to our original substitution $x = \sin\theta$, and correspondingly $\theta = \arcsin x$, we get

$$\begin{aligned}
&= \tfrac{1}{4} \sin(2 \arcsin x) + \tfrac{1}{2} \arcsin x + C \\
&= \tfrac{1}{4} \cdot 2 \sin(\arcsin x) \cos(\arcsin x) + \tfrac{1}{2} \arcsin x + C \\
&= \tfrac{1}{2} \cdot x \cdot \sqrt{1-x^2} + \tfrac{1}{2} \arcsin x + C. \qquad \square
\end{aligned}$$

Assignment 5: due Wednesday, April 11

Section 6.8, Problems 19, 20, 43, 44, 61, and 62. Problems 20, 44, and 62 will be graded carefully.

Find the derivative of $x \ln x$ with respect to x. Bonus: find an antiderivative of $\ln x$. (Hint: take $x \ln x - g(x)$, for some very simple function g.)

This will be graded carefully.

Assignment 6: due Friday, April 13

Section 6.8, Problems 45–48. Problems 46 and 48 will be graded carefully.

Section 7.1, Problems 1–4 and 10–13. The even-numbered problems will be graded carefully.

Section 7.1, Problem 56. This will be discussed in tutorial on Thursday. It will be graded carefully.

Math 133, Lecture 7: Integration by parts

Charles Staats

Wednesday, 11 April 2012

1 The Theory

I told you last lecture that there are two major "general techniques" for integration: u-substitution, which corresponds to the chain rule and which you had already seen, and integration by parts, which corresponds to the product rule and which you had not seen. Today, we will discuss the latter: integration by parts. It is less powerful than the product rule for differentiation, since it is only useful for some products; but it is nevertheless an important tool to have in your arsenal.

Let's start with the product rule for differentiation:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx},$$

or in differential form,

$$d(uv) = u\ dv + v\ du.$$

If we integrate both sides, we get

$$\int d(uv) = \int u\ dv + \int v\ du,$$

i.e.,

$$uv + C_1 = \int u\ dv + \int v\ du.$$

If we rearrange the equation so that there is an indefinite integral on each side, then the constant C_1 can be absorbed into one of the indefinite integrals:

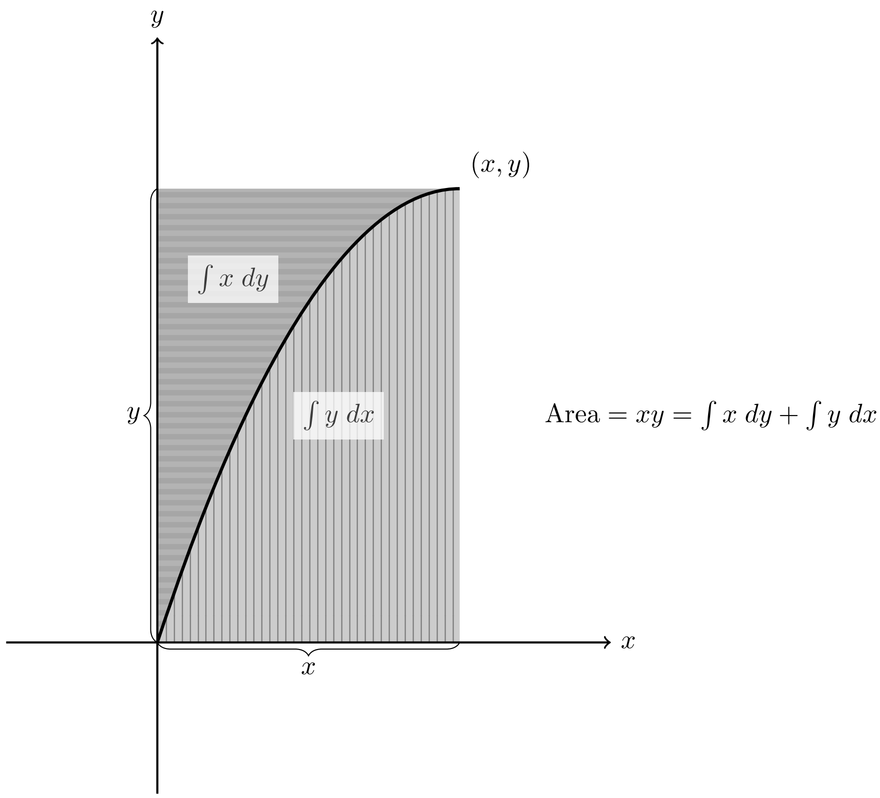
$$uv - \int v\ du = \int u\ dv.$$

This formula, for "integration by parts," is typically used to write one integral, $\int u\,dv$, in terms of another, hopefully easier integral, $\int v\,du$. Just to make this clear, I'll rewrite it as

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

In actual examples, the hardest part tends to be finding a useful way to rewrite an integrand like $x\cos x\,dx$ as $u\,dv$. (In this particular case, the useful choice is $u = x$ and $dv = \cos x\,dx$.)

There's also a picture, in which I've substituted x for u and y for v. Like many of our pictures, it's more than a mnemonic device but less than a proof.



As the picture suggests, we can use integration by parts to handle inverse functions as well as products: If we know how to integrate $y = f(x)$, and want to know how to integrate $x = f^{-1}(y)$, we can take

$$\begin{aligned}\int f^{-1}(y)\ dy &= \int x\ dy\\ &= xy - \int y\ dx\\ &= xy - \int f(x)\ dx.\end{aligned}$$

2 Examples

Example 1. (Example 1, p. 387 in the textbook) Using integration by parts, find

$$\int x \cos x\ dx.$$

Hint: take $u = x$ and $dv = \cos x\ dx$.

Solution. To get $dv = \cos x\ dx$, we need to let v be an antiderivative of $\cos x$, for instance, $v = \sin x$. Then we have

$$\begin{aligned} u &= x & v &= \sin x\\ du &= dx & dv &= \cos x\ dx.\end{aligned}$$

Applying this to the integration by parts formula

$$\int u\ dv = uv - \int v\ du,$$

we obtain

$$\begin{aligned}\int (x)(\cos x\ dx) &= (x)(\sin x) - \int (\sin x)(dx)\\ \int x\cos x\ dx &= x\sin x - \int \sin x\ dx\\ &= x\sin x - (-\cos x) + C\\ &= x\sin x + \cos x + C.\end{aligned}$$

□

Note that, in order to make this work, it was important that $v\ du$ be *simpler* to integrate than $u\ dv$. In this case, $u = x$ was chosen so that passing to $du = dx$ would give something simpler, and with $dv = \cos x\ dx$, we at least have that passing from dv to $v = \sin x$ does not make things any more complicated.

By contrast, if we had attempted to set $u = \cos x$ and $dv = x\ dx$, we'd have that passing from dv to $v = \frac{1}{2}x^2$ actually makes things more complicated. And indeed, integration by parts gives that

$$\begin{aligned}\int x\cos x\;dx &= \int u\;dv \\ &= uv - \int v\;du \\ &= (\cos x)(\tfrac{1}{2}x^2) - \int \tfrac{1}{2}x^2(-\sin x\;dx).\end{aligned}$$

This is *correct*, but not *helpful*: we've expressed the integral we wanted in terms of a more complicated integral that is even harder to integrate.

Exercise 2. Using integration by parts, find

$$\int x\sin x\;dx.$$

Hint: take $u = x$ and $dv = \sin x\;dx$.

Solution. To get $dv = \sin x\;dx$, we need to let v be an antiderivative of $\sin x$, for instance, $v = -\cos x$. Then we have

$$\begin{aligned} u &= x & v &= -\cos x \\ du &= dx & dv &= \sin x\;dx. \end{aligned}$$

Applying this to the integration by parts formula

$$\int u\;dv = uv - \int v\;du,$$

we obtain

$$\begin{aligned}\int (x)(\sin x\;dx) &= (x)(-\cos x) - \int(-\sin x)(dx) \\ \int x\sin x\;dx &= -x\cos x + \int \cos x\;dx \\ &= -x\cos x + \sin x + C.\end{aligned}$$

□

Example 3. Find

$$\int \ln x \; dx$$

via two different approaches: First, considering this as a product; second, by considering $y = \ln x$ to be the inverse function of $x = e^y$. (Both approaches are, in the end, different ways of looking at "integration by parts.")

Assignment 6: due Friday, April 13

Section 6.8, Problems 45–48. Problems 46 and 48 will be graded carefully.

Section 7.1, Problems 1–4 and 10–13. The even-numbered problems will be graded carefully.

Section 7.1, Problem 56. This will be discussed in tutorial on Thursday. It will be graded carefully.

Assignment 7: due Monday, April 16

Section 7.2, Problems 1–8 and 37–38. The even-numbered problems will be graded carefully.

Test some time next week

Tentatively, the test will be on Wednesday and cover everything up to and including integration by parts. I'll try to give more specific information by next lecture about to what extent this includes trig substitution.

Note: you will not be required to SOLVE any differential equations using the techniques of Section 6.6 in the textbook. However, there are many related questions (including other ways of solving differential equations) I could ask you, so please be very careful before assuming a particular type of question will not show up. When in doubt, ask me (in person, in class, and/or by email).

Math 133, Lecture 8: Integrating trig functions

Charles Staats

Friday, 13 April 2012

1 The scalpel and the hammer

We already know how to integrate $\sin nx$ or $\cos nx$. In today's lecture, we will be considering how to integrate more complicated trig functions, like $sin^4 x$ or $\sin 3x \cos 4x$. There are essentially two basic ideas. In many situations, it is possible, by a clever choice of u and du, to use a simple u-substitution rewrite the integral as

$$\int f(u)\, du,$$

where $f(u)$ does not involve any trig functions. This is what I call the "scalpel"—a delicate choice of u allows one to "get rid of" the trig operations altogether.

If a clever u-substitution does not work, it is often possible to use the *product-to-sum formulas* to rewrite the integral as a nicer trig function. As a very basic example, the product-to-sum formulas allow us to rewrite the product

$$\sin x \cos x = \tfrac{1}{2} \sin(2x)$$

as a "sum" (of only one term, in this case). This technique does *not* get rid of the trig functions, but if carried out repeatedly, can convert any product of sines and cosines into a sum that we already know how to integrate. I've called it a "hammer" because it is more powerful than the "scalpel," in that anything that can be done by the first method can, in principle, be done by the second method. But the first method will often give a nicer result.

2 The scalpel: u-substitution for odd exponents

First, let's see an example.

Example 1. Find $\displaystyle\int \sin^2 x \cos x\, dx$.

Solution. Let $u = \sin x$, so that $du = \cos x\, dx$. Then we have

$$\begin{aligned} \int \underbrace{\sin^2 x}_{u^2} \underbrace{\cos x\, dx}_{du} &= \int u^2\, du \\ &= \tfrac{1}{3} u^3 + C \\ &= \tfrac{1}{3} \sin^3 x + C. \end{aligned}$$

□

Note that when we rewrote the integral in terms of u, we completely lost all of the trig functions. We had to re-insert them at the end to put things back in terms of x, but to do the actual integral, all we needed to know was how to integrate a polynomial in u—arguably, the most straightforward type of integral.

More complicated examples often take the form $\sin^n x$, $\cos^n x$, or even $\sin^n x \cos^m x$. The important criterion for such things is this:

Is the exponent (or at least one exponent) odd?

If the answer is "yes", then the scalpel is likely to work, although you will need to remember the following consequences of the Pythagorean identities:

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \cos^2 x &= 1 - \sin^2 x \\ \sin^2 x &= 1 - \cos^2 x. \end{aligned}$$

If the answer is "no," then you probably need to skip to the next section and apply the "hammer" of the product-to-sum formulas.

Example 2. Find $\displaystyle\int \sin^7 x\, dx$.

Solution. Since the exponent of $\sin x$ is odd, we will want to let $du = \sin x\; dx$. Antidifferentiating, we see that an appropriate u-substitution may be $u = -\cos x$. Then we have

$$\begin{aligned} u &= -\cos x \\ du &= \sin x\, dx, \end{aligned}$$

and so

$$
\begin{aligned}
\int \sin^7 x \, dx &= \int \sin^6 x \cdot \sin x \, dx \\
&= \int (\underbrace{\sin^2 x}_{1-\cos^2 x})^3 \cdot \sin x \, dx \\
&= \int \underbrace{(1-\cos^2 x)^3}_{(1-u^2)^3} \underbrace{\sin x \, dx}_{du} \\
&= \int (1-u^2)^3 \, du \\
&= \int (1 - 3u^2 + 3u^4 - u^6) \, du \\
&= u - u^3 + \tfrac{3}{5}u^5 - \tfrac{1}{7}u^7 + C \\
&= -\cos x + \cos^3 x - \tfrac{3}{5}\cos^5 x + \tfrac{1}{7}\cos^7 x + C. \qquad \square
\end{aligned}
$$

Exercise 3. Find $\int \cos^7 x \, dx$.

Solution. Since the exponent of $\cos x$ is odd, we will want to let $du = \cos x \, dx$. Antidifferentiating, we see that an appropriate u-substitution may be $u = \sin x$. Then we have

$$\begin{aligned} u &= \sin x \\ du &= \cos x \, dx, \end{aligned}$$

and so

$$\begin{aligned} \int \cos^7 x \, dx &= \int \cos^6 x \cdot \cos x \, dx \\ &= \int (\underbrace{\cos^2 x}_{1-\sin^2 x})^3 \cdot \cos x \, dx \\ &= \int \underbrace{(1 - \sin^2 x)^3}_{(1-u^2)^3} \underbrace{\cos x \, dx}_{du} \\ &= \int (1 - u^2)^3 \, du \\ &= \int (1 - 3u^2 + 3u^4 - u^6) \, du \\ &= u - u^3 + \tfrac{3}{5}u^5 - \tfrac{1}{7}u^7 + C \\ &= \sin x - \sin^3 x + \tfrac{3}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C. \end{aligned}$$ □

I'll let the tutors go over examples of the form $\sin^n x \cos^m x$ where at least one of n, m is odd.

3 The hammer: the product-to-sum formulas

There are three basic trig formulas that you should memorize:

Product-to-Sum Formulas:

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \quad (1)$$

$$\sin\alpha\sin\beta = -\tfrac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)] \quad (2)$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] \quad (3)$$

As the name suggests, these formulas allow you to convert a product of trig functions to a sum of (other) trig functions. They can be very useful when the "scalpel" does not apply (or for that matter, when it does, if you prefer this approach).

Example 4. Find $\int \sin^4 x\, dx$.

Solution. Since the exponent is even, the "scalpel" will not work. However, the product-to-sum formulas will, if we apply them repeatedly.

$$\begin{aligned}
\sin^2 x &= \sin x \sin x \\
&= -\tfrac{1}{2}[\cos(x+x) - \cos(x-x)] \\
&= -\tfrac{1}{2}[\cos 2x - \cos 0] \\
&= -\tfrac{1}{2}(\cos 2x - 1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sin^4 x &= \left(\sin^2 x\right)^2 \\
&= \left[-\tfrac{1}{2}(\cos 2x - 1)\right]^2 \\
&= \tfrac{1}{4}(\cos 2x - 1)^2 \\
&= \tfrac{1}{4}(\cos^2 2x - 2\cos 2x + 1) \\
&= \tfrac{1}{4}\cos^2 2x - \tfrac{1}{2}\cos 2x + \tfrac{1}{4}.
\end{aligned}$$

Finally, we use the product-to-sum formulas again to deal with the $\cos^2 2x$ term:

$$\begin{aligned}
\cos^2 2x &= \cos 2x \cos 2x \\
&= \tfrac{1}{2}[\cos(2x+2x) + \cos(2x-2x)] \\
&= \tfrac{1}{2}[\cos 4x + \cos 0] \\
&= \tfrac{1}{2}[\cos 4x + 1].
\end{aligned}$$

Hence,

$$\begin{aligned}
\sin^4 x &= \tfrac{1}{4}\cos^2 2x - \tfrac{1}{2}\cos 2x + \tfrac{1}{4} \\
&= \tfrac{1}{4}\cdot\tfrac{1}{2}[\cos 4x + 1] - \tfrac{1}{2}\cos 2x + \tfrac{1}{4} \\
&= \tfrac{1}{8}\cos 4x + \tfrac{1}{8} - \tfrac{1}{2}\cos 2x + \tfrac{1}{4} \\
&= \tfrac{1}{8}\cos 4x - \tfrac{1}{2}\cos 2x + \tfrac{3}{8}.
\end{aligned}$$

Now, at long last, we're ready to do the integral:

$$\begin{aligned}
\int \sin^4 x\, dx &= \frac{1}{8}\int \cos \underbrace{4x}_{u} \underbrace{dx}_{\frac{1}{4}du} - \frac{1}{2}\int \cos \underbrace{2x}_{v} \underbrace{dx}_{\frac{1}{2}dv} + \frac{3}{8}\int dx \\
&= \frac{1}{32}\int \cos u\, du - \frac{1}{4}\int \cos v\, dv + \frac{3}{8}\int dx \\
&= \tfrac{1}{32}\sin u - \tfrac{1}{4}\sin v + \tfrac{3}{8}x + C \\
&= \tfrac{1}{32}\sin 4x - \tfrac{1}{4}\sin 2x + \tfrac{3}{8}x + C. \qquad \square
\end{aligned}$$

Given the length of the above solution, perhaps you can see why the "scalpel" method is preferable, when it is available.

Example 5. Find $\int \sin 3x \cos 2x \; dx$.

Solution.

$$\begin{aligned}
\int \sin 3x \cos 2x \; dx &= \int \tfrac{1}{2}[\sin(3x+2x) - \sin(3x-2x)] \; dx \\
&= \frac{1}{2}\int \sin \underbrace{5x}_{u} \underbrace{dx}_{\frac{1}{5}du} - \frac{1}{2}\int \sin x \; dx \\
&= \frac{1}{10}\int \sin u \; du - \frac{1}{2}\int \sin x \; dx \\
&= -\tfrac{1}{10}\cos u + \tfrac{1}{2}\cos x + C \\
&= -\tfrac{1}{10}\cos 5x + \tfrac{1}{2}\cos x + C. \qquad \square
\end{aligned}$$

4 When different methods give different-looking answers

The extensive collection of trig identities often allow one to show that completely different-looking expressions are in fact equal. Many of you may have seen this in exercises like the following:

Exercise 6. (Exercise 12(a), Section 0.7, page 48) Verify that

$$\cos 3t = 4\cos^3 t - 3\cos t.$$

Sometimes, when the same trig integral is done using different methods, we end up with different-looking answers that are (non-obviously) equal up to a constant.

Assignment 7: due Monday, April 16

Section 7.2, Problems 1–8 and 37–38. The even-numbered problems will be graded carefully.

Test Wednesday, April 18

The test will cover everything up to and including integration by parts. You may be required to perform the integrals $\int \frac{dx}{1+x^2}$ and/or $\int \frac{dx}{\sqrt{1-x^2}}$. I recommend that you do these via trig substitution, since this will allow you to receive partial credit, and since studying this technique will help you on future tests also. However, since the book does include these integrals in its list of seventeen "standard integral forms," you will receive full credit for simply memorizing the answer (correctly). Incorrect memorization will receive no credit.

Note: you will not be required to SOLVE any differential equations using the techniques of Section 6.6 in the textbook. However, there are many related questions (including other ways of solving differential equations) I could ask you, so please be very careful before assuming a particular type of question will not show up. When in doubt, ask me (in person, in class, and/or by email).

Assignment 8: due Friday, April 20

Section 7.2, Problems 43 and 44. Problem 44 will be graded carefully.

Section 7.3, Problems 1 and 3.

Section 7.4, Problems 9 and 10. These will be discussed in tutorial on Thursday.

Verify the product-to-sum identities (from Lecture 8) using the more familiar angle-sum and angle-difference formulas.

Math 133, Lecture 9: Trig substitution

Charles Staats

Monday, 16 April 2012

1 Test

Please note that there will be a test on Wednesday, April 18 (i.e., next class meeting). The test will cover everything up to and including integration by parts. You may be required to perform the integrals $\int \frac{dx}{1+x^2}$ and/or $\int \frac{dx}{\sqrt{1-x^2}}$. I recommend that you do these via trig substitution, since this will allow you to receive partial credit, and since studying this technique will help you on future tests also. However, since the book does include these integrals in its list of seventeen "standard integral forms," you will receive full credit for simply memorizing the answer (correctly). Incorrect memorization will receive no credit.

Note: you will not be required to SOLVE any differential equations using the techniques of Section 6.6 in the textbook. However, there are many related questions (including other ways of solving differential equations) I could ask you, so please be very careful before assuming a particular type of question will not show up. When in doubt, ask me (in person, in class, and/or by email).

2 Trig substitution

Recall the following versions of the Pythagorean identities:

$$\begin{aligned} 1 - \sin^2\theta &= \cos^2\theta \\ 1 + \tan^2\theta &= \sec^2\theta \\ \sec^2\theta - 1 &= \tan^2\theta. \end{aligned}$$

In all three of these cases, we have something on the left that is not (obviously) a square, and something on the right that is a square. Thus, if we are asked to take the square root of the thing on the left, we might be well-advised to convert it to the thing on the right. This inspires the technique of trigonometric substitution, a "trick" that is used to figure out a clever u-substitution to get rid of an unfortunate square root in an integral. Here are the general guidelines:

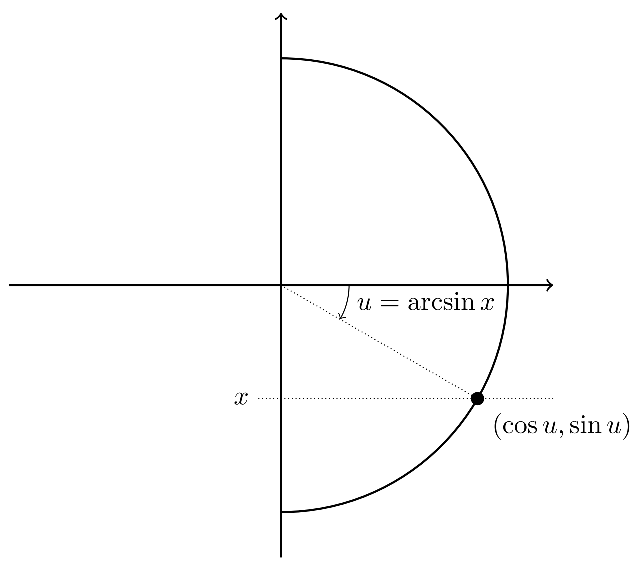
1. To get rid of a radical of the form $\sqrt{a^2 - x^2}$, with $a > 0$, substitute

$$\begin{aligned} u &= \arcsin\left(\frac{x}{a}\right) \\ \frac{x}{a} &= \sin u \\ x &= a \sin u \\ dx &= -a \cos u \; du. \end{aligned}$$

This will have the effect that

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 u} = |a|\sqrt{1 - \sin^2 u} = a\sqrt{\cos^2 u} = a \cos u,$$

where $|a| = a$ since $a > 0$ and $|\cos u| = \cos u$ since, by definition of the arcsin function, $\cos u \geq 0$. The arcsin function always takes angles in the right half of the unit circle:



2. To get rid of a radical of the form $\sqrt{a^2 + x^2}$, with $a > 0$, substitute

$$\begin{aligned} u &= \arctan\left(\frac{x}{a}\right) \\ \frac{x}{a} &= \tan u \\ x &= a \tan u \\ dx &= a \sec^2 u \; du. \end{aligned}$$

This will have the effect that

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 u} = a\sqrt{1 + \tan^2 u} = a\sqrt{\sec^2 u} = a \sec u.$$

Note that, once again, we know that $\sec u = \frac{1}{\cos u}$ is positive, because $u = \arctan x$, and like the arcsine function, the arctangent function always takes angles in the right half of the unit circle, where the cosine is positive.

3. To get rid of a radical of the form $\sqrt{x^2 - a^2}$, substitute

$$\begin{aligned} u &= \operatorname{arcsec}\left(\frac{x}{a}\right) = \arccos\left(\frac{a}{x}\right) \\ \frac{a}{x} &= \cos u \\ \frac{x}{a} &= \sec u \\ x &= a \sec u \\ dx &= a \sec u \tan u \; du. \end{aligned}$$

This will have the effect that

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 u - a^2} = a\sqrt{\sec^2 u - 1} = a\sqrt{\tan^2 u} = \pm a \tan u.$$

Unfortunately, we cannot really get rid of the absolute value here, since $\tan u$ represents the slope of a line with angle u, and both positive and negative slopes are found in any semicircle of the unit circle.

All of this was a bit long; the part that you really need to remember to do the actual integrals is summarized in this table, which is more or less copied from p. 400 of the textbook:

	If you see	Sustitute
1.	$a^2 - x^2$	$x = a \sin u$
2.	$a^2 + x^2$	$x = a \tan u$
3.	$x^2 - a^2$	$x = a \sec u$

3 Examples

Our two examples might well show up on the test Wednesday. They do not really require trig substitution, but I recommend you learn how to do them this way rather than by rote memorization.

Example 1. Find $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}}$.

Solution. Seeing an inconvenient occurrence of $1 - x^2$, we substitute

$$\begin{aligned} x &= \sin u \\ dx &= \cos u \; du. \end{aligned}$$

Our integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{\cos u \; du}{\sqrt{1-\sin^2 u}} \\ &= \int \frac{\cos u \; du}{\sqrt{\cos^2 u}} \\ &= \int \frac{\cos u \; du}{\cos u},\end{aligned}$$

since we're really taking $u = \arcsin x$, so $\cos u \geq 0$

$$\begin{aligned}&= \int du \\ &= u + C \\ &= \arcsin x + C.\end{aligned}$$

□

Example 2. Find $\displaystyle\int \frac{dx}{1+x^2}$.

Solution. Seeing an inconvenient occurrence of $1+x^2$, we substitute

$$\begin{aligned}x &= \tan u \\ dx &= \sec^2 u \; du.\end{aligned}$$

Our integral becomes

$$\begin{aligned}\int \frac{dx}{1+x^2} &= \int \frac{\sec^2 u \; du}{1+\tan^2 u} \\ &= \int \frac{\sec^2 u \; du}{\sec^2 u} \\ &= \int du \\ &= u + C \\ &= \arctan x + C.\end{aligned}$$

□

Now, for some "more serious" examples.

Example 3. Find $\displaystyle\int x\sqrt{4-x^2}\; dx$.

Solution. This is a "trick question." You don't need trig substitution, since a simpler u-substitution will do: Set

$$\begin{aligned}u &= 4 - x^2 \\ du &= -2x \; dx \\ x \; dx &= -\tfrac{1}{2} du.\end{aligned}$$

Then we have

$$\begin{aligned}\int x\sqrt{4-x^2}\,dx &= -\frac{1}{2}\int \sqrt{u}\,du\\ &= -\frac{1}{2}\int u^{1/2}\,du\\ &= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C\\ &= -\frac{1}{3}(4-x^2)^{3/2}+C. \qquad\square\end{aligned}$$

The previous example is meant to underscore an important principle:

> *Never use trig substitution until you've checked to see if a more basic technique will work.*

However, sometimes it is necessary.

Example 4. Find $\int x^2\sqrt{4-x^2}\,dx$.

Solution. Seeing an inconvenient $4-x^2 = 2^2-x^2$, with no obvious way to get rid of it by a u-substitution, we use trig substitution:

$$\begin{aligned}u &= \arcsin\left(\frac{x}{2}\right)\\ x &= 2\sin u\\ dx &= 2\cos u\,du.\end{aligned}$$

Then we have

$$\begin{aligned}\int x^2\sqrt{4-x^2}\,dx &= \int 4\sin^2 u\sqrt{4-4\sin^2 u}\cdot 2\cos u\,du\\ &= \int 4\sin^2 u\cdot 2\cos u\cdot 2\cos u\,du\\ &= 16\int \sin^2 u\,\cos^2 u\,du.\end{aligned}$$

The product-to-sum formulas show us that

$$\begin{aligned}\sin u\cdot\sin u &= -\tfrac{1}{2}\cos(u+u)+\tfrac{1}{2}\cos(u-u)\\ &= -\tfrac{1}{2}\cos 2u+\tfrac{1}{2}\\ \cos u\cdot\cos u &= \tfrac{1}{2}\cos(u+u)+\tfrac{1}{2}\cos(u-u)\\ &= \tfrac{1}{2}\cos 2u+\tfrac{1}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}
\int x^2\sqrt{4-x^2}\,dx &= 16\int \left(-\tfrac{1}{2}\cos 2u+\tfrac{1}{2}\right)\left(\tfrac{1}{2}\cos 2u+\tfrac{1}{2}\right)\,du \\
&= 4\int (1-\cos 2u)(1+\cos 2u)\,du \\
&= 4\int (1-\cos^2 2u)\,du \\
&= 4\int du - 4\int \cos^2 \underbrace{2u}_{v}\underbrace{du}_{\frac{1}{2}dv} \\
&= 4\int du - 2\int \cos^2 v\,dv.
\end{aligned}$$

As above,

$$\cos^2 v = \tfrac{1}{2}\cos 2v + \tfrac{1}{2},$$

so

$$\begin{aligned}
\int x^2\sqrt{4-x^2}\,dx &= 4\int du - \int (\cos 2v + 1)\,dv \\
&= 4\int du - \int \cos \underbrace{2v}_{w}\underbrace{dv}_{\frac{1}{2}dw} - \int dv \\
&= 4\int du - \int dv - \frac{1}{2}\int \cos w\,dw \\
&= 4u - v - \tfrac{1}{2}\sin w + C \\
&= 4u - v - \tfrac{1}{2}\sin 2v + C \\
&= 4u - 2u - \tfrac{1}{2}\sin 4u + C \\
&= 2u - \tfrac{1}{2}\sin 4u + C \\
&= 2\arcsin\left(\frac{x}{2}\right) - \frac{1}{2}\sin\left(4\arcsin\left(\frac{x}{2}\right)\right) + C
\end{aligned}$$

To finish the problem, one should find an algebraic expression for $\sin(4\arcsin\theta)$, but I somehow suspect we'll be out of time at this point. $\square$

Assignment 7: due Monday, April 16

Section 7.2, Problems 1–8 and 37–38. The even-numbered problems will be graded carefully.

Test Wednesday, April 18

The test will cover everything up to and including integration by parts. You may be required to perform the integrals $\int \frac{dx}{1+x^2}$ and/or $\int \frac{dx}{\sqrt{1-x^2}}$. I recommend that you do these via trig substitution, since this will allow you to receive partial credit, and since studying this technique will help you on future tests also. However, since the book does include these integrals in its list of seventeen "standard integral forms," you will receive full credit for simply memorizing the answer (correctly). Incorrect memorization will receive no credit.

Note: you will not be required to SOLVE any differential equations using the techniques of Section 6.6 in the textbook. However, there are many related questions (including other ways of solving differential equations) I could ask you, so please be very careful before assuming a particular type of question will not show up. When in doubt, ask me (in person, in class, and/or by email).

Assignment 8: due Friday, April 20

Section 7.2, Problems 43 and 44. Problem 44 will be graded carefully.

Section 7.3, Problems 1 and 3.

Section 7.4, Problems 9 and 10. These will be discussed in tutorial on Thursday.

Verify the product-to-sum identities (from Lecture 8) using the more familiar angle-sum and angle-difference formulas.

Math 133, Lecture 10: l'Hôpital's Rule

Charles Staats

Friday, 20 April 2012

1 A couple notes from the test

Although I have not yet finished grading the tests, I have observed a couple errors that should be corrected.

1. Don't use the same letter to mean two different things. For instance, I had more than one person write something like

$$\xcancel{\begin{aligned} x &= \tan x \\ dx &= \sec^2 x \, dx \end{aligned}}$$

   This is incredibly confusing. Apart from other considerations, mixing up letters like this, while understandable in the heat of the moment, makes it very difficult for me as grader to tell what you were thinking, which makes it hard to give you partial credit. If you have not already used u for something else, you might write something like

$$\begin{aligned} x &= \tan u \\ dx &= \sec^2 u \, du \end{aligned}$$

   for the above. If you have already used u, then you need to be creative and come up with a different symbol. You might consider v, w, t, z, or for a trig substitution like this one, θ or ϕ.

2. When you do integration by parts, remember that you should use a *minus* sign rather than a *plus* sign.

$$\int u \, dv = uv \overset{\downarrow}{-} \int v \, du.$$

2 Trig substitution: wrap-up

First of all, I'd like to note that to use trig substitution facilely, you may want to memorize the following formulas, which I have so far avoided using more than necessary:

$$\boxed{\begin{aligned} &\frac{d}{dx}\sec x = \sec x \tan x \\ &\frac{d}{dx}\tan x = \sec^2 x \\ &\int \sec u \, du = \ln|\sec u + \tan u| + C. \end{aligned}}$$

The last formula, for $\int \sec u\, du$, you should have derived on a homework exercise (Section 7.1, Problem 56).

As I final example, let's do Example 5 from p. 401 of the textbook.

Example 1. (Example 5, p. 401) Find $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$.

Solution. First, recall the table from the previous lecture:

	If you see	Sustitute
1.	$a^2 - x^2$	$x = a\sin u$
2.	$a^2 + x^2$	$x = a\tan u$
3.	$x^2 - a^2$	$x = a\sec u$

In our case, the unfortunate thing we are trying to simplify is $\sqrt{9+x^2}$, which has the second form: $3^2 + x^2$. Thus, we should substitute

$$\begin{aligned} u &= \arctan(x/3) \\ x &= 3\tan u \\ dx &= 3\sec^2 u \, du. \end{aligned}$$

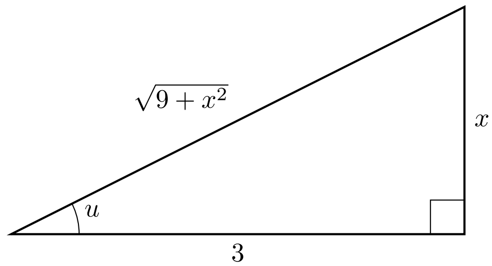
The last comes from the derivative for the tangent function, which I have just asked you to memorize. Now, this substitution gives

$$\begin{aligned} \int \frac{dx}{\sqrt{9+x^2}} &= \int \frac{3\sec^2 u\, du}{\sqrt{9+9\tan^2 u}} \\ &= \int \frac{3\sec^2 u\, du}{3\sqrt{1+\tan^2 u}} \\ &= \int \frac{3\sec^2 u\, du}{3\sqrt{\sec^2 u}} \\ &= \int \frac{\sec^2 u\, du}{\sec u}, \end{aligned}$$

where we are using the fact that when $u = \arctan x$, u is an angle in the right half of the unit circle, and so $\cos u$ (and its reciprocal $\sec u$) are both positive

$$\begin{aligned} &= \int \sec u\, du \\ &= \ln|\sec u + \tan u| + C. \end{aligned}$$

Now, by definition, $\tan u = x/3$. Looking at the right triangle



we see that

$$\sec u = \frac{1}{\cos u} = \frac{\sqrt{9+x^2}}{3},$$

and so we obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{9+x^2}} &= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C \\ &= \ln\frac{|\sqrt{9+x^2}+x|}{|3|} + C \\ &= \ln\left|\sqrt{9+x^2}+x\right| - \ln 3 + C \\ &= \ln\left|\sqrt{9+x^2}+x\right| + C_1,\end{aligned}$$

where $C_1 = C - \ln 3$. □

3 Using derivatives to compute limits: l'Hôpital's Rule

If you recall the so-called "Main Limit Theorem" from Math 131, one of the important parts of it was that "limits distribute over quotients":

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$$

whenever the expression on the right makes sense.

If the expression on the right does not make sense—for instance, if the limits on the right come down to 0/0 or ∞/∞—then our previous method was to try some fancy re-arrangement. Now, we are going to introduce a more direct method for dealing with such limits.

Theorem. Suppose that

$$\frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$$

is of the form $0/0$ or ∞/∞ (and, in particular, the Main Limit Theorem does not apply naïvely). Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)},$$

provided that the expression on the right makes sense (i.e., f' and g' are defined, and their ratio has a limit).

An example, to show the usefulness of this rule:

Example 2. Compute $\lim_{x\to 0} \frac{\sin 3x}{5x}$.

Solution. First, we compute the limits of the numerator and denominator, using continuity:

$$\begin{aligned} \lim_{x\to 0} \sin 3x &= \sin(3\cdot 0) = 0 \\ \lim_{x\to 0} 5x &= 5\cdot 0 = 0. \end{aligned}$$

Thus, this is of the form $0/0$, and so it is legal to attempt l'Hôpital's rule:

$$\begin{aligned} \lim_{x\to 0} \frac{\sin 3x}{5x} &= \lim_{x\to 0} \frac{3\cos 3x}{5} && \text{by l'Hôpital} \\ &= \tfrac{3}{5}\cos(3\cdot 0) \\ &= \tfrac{3}{5}. \end{aligned}$$

You may recall that proving this sort of limit was rather trying to evaluate back when we first discussed it at the beginning of last quarter. □

Warning. *You may be tempted to apply l'Hôpital's rule whenever you have a fraction to take a limit of. This is, most emphatically,* WRONG. *You can only apply l'Hôpital's rule after first checking to make sure that your limit is of the form $0/0$ or ∞/∞.*

Example 3. (Example 5, p. 425) Find $\lim_{x\to 0} \frac{1-\cos x}{x^2+3x}$.

Wrong solution.

$$\lim_{x\to 0} \frac{1-\cos x}{x^2+3x} = \lim_{x\to 0} \frac{\sin x}{2x+3} \overset{\downarrow}{=} \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

This mistaken solution includes two uses of l'Hôpital's rule. The first is correct, since the expression is of the form $0/0$. The second is wrong. □

Correct solution. Note that

$$\begin{aligned} 1 - \cos 0 &= 1 - 1 = 0 \\ 0^2 + 3(0) &= 0. \end{aligned}$$

Thus, we may apply l'Hôpital's rule to obtain

$$\lim_{x\to 0} \frac{1-\cos x}{x^2+3x} = \lim_{x\to 0} \frac{\sin x}{2x+3}.$$

At this point, l'Hôpitalis no longer needed: the Main Limit Theorem, plus continuity, suffices.

$$= \frac{\sin 0}{2\cdot 0+3} = \frac{0}{3} = 0. \qquad \square$$

In a sense, the cardinal sin in the the wrong solution was to apply l'Hôpital's rule when the main limit theorem applied (and hence l'Hôpital was illegal). In spite of its ease of use, you always need to remember that l'Hôpital is a "backup" tool, to be used when the more direct Main Limit Theorem "breaks." If it ain't broke, don't fix it.

Assignment 9: due Monday, April 23

Section 7.3, Problems 2, 4, 5, and 6. Problems 4 and 6 will be graded carefully.

Section 7.4, Problems 11–14. Problems 12 and 14 will be graded carefully.

Section 8.1, Problems 1 and 2.

Assignment 10: due Wednesday, April 25

Section 7.3, Problems 13 and 14. Problem 14 will be graded carefully.

Section 7.4, Problems 17 and 18. You may want to look at Example 7 on p. 402. Problem 18 will be graded carefully.

Section 8.1, Problems 3–6. Problems 4 and 6 will be graded carefully.

Section 8.2, Problems 1 and 2.

Recall the following problem from Assignment 1:

> A clown starts out with a huge bucket of 5000 identical tiny blue marbles. Once every thirty seconds, he randomly removes a marble. If it is blue, he replaces it by an identical red marble; if it is red, he does not replace it by anything, and the total number of marbles decreases. Write a pair of differential equations that describes the numbers R of red marbles and B of blue marbles over time.

Without solving the differential equations, use l'Hôpital's rule to compute

$$\lim_{t \to t_0} \frac{R}{B}$$

at the one time t_0 where l'Hôpital's rule applies to this limit. Also find this time t_0. This problem will be discussed in tutorial on Tuesday. It will be graded carefully.

[Note: this limit will have no physical meaning, since in reality, you can only have an integer number of balls.]

Math 133, Lecture 11: Indeterminate forms

Charles Staats

Monday, 23 April 2012

1 Indeterminate forms

In considering how a function $f(t)$ behaves as t approaches a limit, it is often useful to think in terms of "forces" battling over control of the function. For instance, if we look at

$$\lim_{t\to\infty} \frac{t^2+1}{e^{-t}},$$

the top function approaches infinity while the bottom function approaches zero. Thus, we might say this represents the "form" $\infty/0^+$. (The "plus" sign in 0^+ indicates that the denominator is always positive.) As the numerator gets really big, it represents a "force" trying to push the function to ∞. The denominator is getting really small; since dividing by a very small number gives something very big, this "force" is also trying to push the overall function to infinity. Thus, there is no mystery here: both the "forces" are trying to push the function to infinity, so

$$\infty/0^+ = \infty.$$

Things get interesting when you have two competing "forces." For instance, consider instead

$$\lim_{t\to\infty} \frac{t^2+1}{e^{t}}.$$

In this case, the "form" is ∞/∞. The two "forces" are in conflict: you are taking a numerator that is getting really big (trying to push the function to ∞) and dividing it by a denominator that is also getting really big (hence trying to push the function to 0). Thus, ∞/∞ is considered an "indeterminate form," because knowing the form alone does not determine the limit: you need to understand the relative "strength" of the two competing "forces."

For the indeterminate forms ∞/∞ and $0/0$, we have already discussed a powerful tool that can often be applied: l'Hôpital's rule.

Exercise 1. Find $\lim_{t\to\infty} \frac{t^2+1}{e^t}$.

Solution. $\square$

There are seven "standard indeterminate forms":

$$0/0, \quad \infty/\infty, \quad 0\cdot\infty, \quad \infty-\infty, \quad 0^0, \quad \infty^0, \quad \text{and } 1^\infty.$$

Only the first two can be solved by *direct* applications of l'Hôpital's rule. However, it is often possible to transform the others into the first two.

Example 2. $(0\cdot\infty)$ Find $\lim_{x\to 0^+} x\ln x$.

Solution. As $x \to 0^+$, we see that $x \to 0$ and $\ln x \to -\infty$. Thus, this follows the indeterminate form $0\cdot\infty$ (or, if you want to keep track of signs, $0^+\cdot(-\infty)$). Since it is not a fraction, we can't apply l'Hôpital's rule directly. But it is easy enough to make this into a fraction:

$$\begin{aligned}
\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} && \left(\frac{\infty}{\infty}\right)\\
&= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} && \text{l'Hôpital}\\
&= \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}}\\
&= \lim_{x\to 0^+} -x^{-1+2}\\
&= \lim_{x\to 0^+} -x\\
&= 0. && \square
\end{aligned}$$

Example 3. $(\infty-\infty)$ (Example 6, p. 431 in the textbook) Find

$$\lim_{x\to 1^+}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right).$$

Solution. Both the left and the right term go to infinity as $x \to 1^+$, so this an instance of the indeterminate form $\infty - \infty$. We combine the fractions:

$$\begin{aligned}
\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) &= \lim_{x\to 1^+}\frac{x\ln x - 1\cdot(x-1)}{(x-1)\ln x} \\
&= \lim_{x\to 1^+}\frac{x\ln x - x + 1}{(x-1)\ln x} \qquad \left(\frac{0}{0}\right).
\end{aligned}$$

The rest of the solution is left as an in-class exercise.

□

The three remaining indeterminate forms are the "exponential forms": 0^0, ∞^0, and 1^∞. The same trick typically works for all three: evaluate the *logarithm* of the limit, rather than the limit itself. This transforms these forms into versions of the indeterminate form $0\cdot\infty$ (ignoring signs):

$$\begin{aligned}
\ln 0^0 &= 0\ln 0 = 0\cdot(-\infty) \\
\ln \infty^0 &= 0\ln\infty = 0\cdot\infty \\
\ln 1^\infty &= \infty\ln 1 = \infty\cdot 0.
\end{aligned}$$

We'll just do one example.

Example 4. Find $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution. Note that this is of the form 1^∞. Let

$$y = \left(1+\frac{1}{x}\right)^x.$$

Then we have

$$
\begin{aligned}
\ln y &= x \ln\left(1 + x^{-1}\right) \\
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} x \ln(1 + x^{-1}) && (\infty \cdot 0) \\
&= \lim_{x\to\infty} \frac{\ln(1 + x^{-1})}{1/x} && \left(\frac{0}{0}\right) \\
&= \\
&\vdots \\
&\vdots \\
&\vdots \\
&= 1.
\end{aligned}
$$

Warning. *When you take the limit of the* logarithm *of the thing you want, do not forget to get rid of the logarithm afterwards by exponentiating.*

Consequently,

$$
\begin{aligned}
\lim_{x\to\infty} y &= \lim_{x\to\infty} \exp(\ln y) \\
&= \exp\left(\lim_{x\to\infty} \ln y\right) && \text{since } \exp(z) = e^z \text{ is continuous} \\
&= \exp(1) \\
&= e.
\end{aligned}
$$

□

2 Why does l'Hôpital's rule work?

I'm running out of time in typing this up. In any case, I have the feeling that my first attempt to explain this will be terribly confusing, so I'd rather not have it committed to paper. But please, please ask questions here; I find that I am able to produce much more satisfying explanations of things when I have your questions to inspire me.

Assignment 10: due Wednesday, April 25

Section 7.3, Problems 13 and 14. Problem 14 will be graded carefully.

Section 7.4, Problems 17 and 18. You may want to look at Example 7 on p. 402. Problem 18 will be graded carefully.

Section 8.1, Problems 3–6. Problems 4 and 6 will be graded carefully.

Section 8.2, Problems 1 and 2.

Recall the following problem from Assignment 1:

> A clown starts out with a huge bucket of 5000 identical tiny blue marbles. Once every thirty seconds, he randomly removes a marble. If it is blue, he replaces it by an identical red marble; if it is red, he does not replace it by anything, and the total number of marbles decreases. Write a pair of differential equations that describes the numbers R of red marbles and B of blue marbles over time.

Without solving the differential equations, use l'Hôpital's rule to compute

$$\lim_{t \to t_0} \frac{R}{B}$$

at the one time t_0 where l'Hôpital's rule applies to this limit. Also find this one special time t_0 where l'Hôpital's rule works. This problem will be discussed in tutorial on Tuesday. It will be graded carefully.

[Note: this limit will have no physical meaning, since in reality, you can only have an integer number of balls.]

Assignment 11: due Friday, April 27

Section 7.4, Problem 29. This will be graded carefully. It will be discussed in tutorial on Thursday.

Section 8.1, Problems 9 and 10. Problem 10 will be graded carefully.

Section 8.2, Problems 3–6, 19, and 21. Problems 4, 6, 19, and 21 will be graded carefully.

Math 133, Lecture 12: Improper integrals

Charles Staats

Wednesday, 25 April 2012

1 Integrals over unbounded x-values

Today, we will discuss an apparent paradox: infinite regions with finite area. In so doing, we will give our first theoretical extension of the definition of the integral. Fortunately, this should be a good deal simpler and faster than our initial definition of the integral using Riemann sums.

From a technical perspective, actually computing these "improper integrals" often requires a combination of skills we have so far discussed separately: finding integrals, and computing limits (often with l'Hôpital's rule).

Here's the first conceptual hurdle: $\int_a^b f(x)\,dx$ is a function of a and b. If you find this somewhat confusing, perhaps you will find an example less so.

Exercise 1. Find $\displaystyle\int_a^b xe^{-x}\,dx$.

Solution. First, find the indefinite integral, using integration by parts. To find $\int xe^{-x}\,dx$, take

$$\begin{aligned} u &= x & v &= -e^{-x} \\ du &= dx & dv &= e^{-x}\,dx. \end{aligned}$$

Then we have

$$\begin{aligned} \int xe^{-x}\,dx &= -xe^{-x} - \int (-e^{-x})\,dx \\ &= -xe^{-x} - \int e^{-x}(-dx) \\ &= -xe^{-x} - e^{-x} + C. \end{aligned}$$

Now, substitute in a and b, as usual for finding a definite integral:

$$\begin{aligned} \int_a^b xe^{-x}\,dx &= \left[-xe^{-x} - e^{-x}\right]_{x=a}^{b} \\ &= (-be^{-b} - e^{-b}) - (-ae^{-a} - e^{-a}) \\ &= -be^{-b} - e^{-b} + ae^{-a} + e^{-a}. \end{aligned}$$

□

Definition. We will define notation as follows:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

Having defined these, we use them to define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx.$$

These integrals are called *improper integrals*.

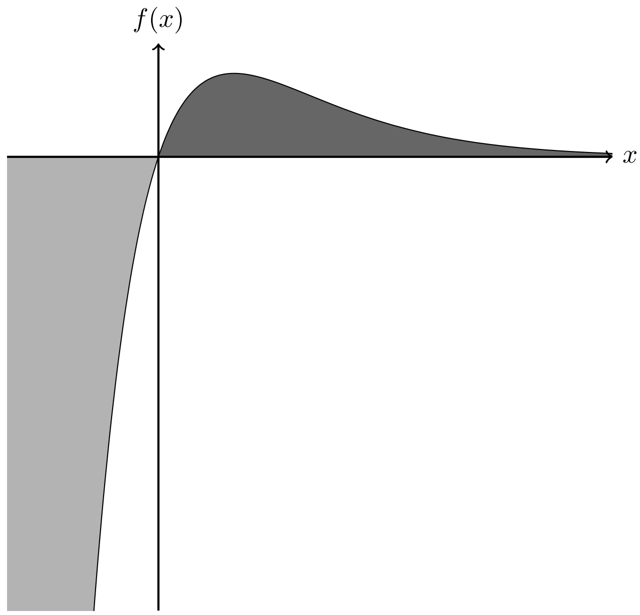
Exercise 2. Find $\displaystyle\int_0^\infty xe^{-x}\,dx$.

Solution. By definition,

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x}\,dx \\
&= \lim_{b\to\infty} -be^{-b} - e^{-b} + 0e^{-0} + e^{-0} \\
&= \left(\lim_{b\to\infty} -be^{-b}\right) - 0 + 0 + 1 \\
&= \left(\lim_{b\to\infty} \frac{-b}{e^b}\right) + 1 && \left(\frac{\infty}{\infty}\right) \\
&= \left(\lim_{b\to\infty} \frac{-1}{e^b}\right) + 1 \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$

□

What do "improper integrals" like these really mean? The function $f(x) = xe^{-x}$ looks like this:



The integral $\int_0^\infty xe^{-x}\,dx$ is the area of the dark shaded region, that is, the area under the curve over the entire infinite interval $[0, \infty)$. Looking at it, you may find it plausible that this area is finite, even though the interval is infinite.

On the other hand, the integral $\int_{-\infty}^0 xe^{-x}\,dx$ represents the (negative of)[1] the area of the light gray region. Although I will not do the calculation in an effort to save time, you should find it plausible enough that this integral is $-\infty$. Likewise,

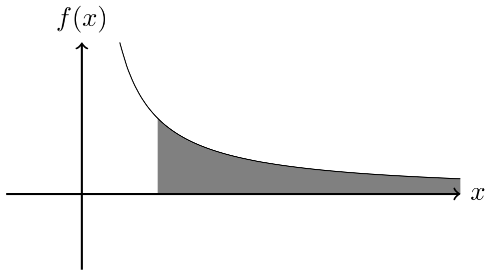
$$\begin{aligned}\int_{-\infty}^{\infty} xe^{-x}\,dx &= \int_{-\infty}^{0} xe^{-x} + \int_{0}^{\infty} xe^{-x}\,dx \\ &= (-\infty) + 1 \\ &= -\infty.\end{aligned}$$

This brings us to another definition:

Definition. An improper integral is said to *converge* if the limit defining it exists and is finite.

Just because an integral "looks" like the region might have finite area, does not necessarily mean it does. Consider the improper integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$. The region looks like this:

[1]Since this area is *under* the x-axis, the integral is negative.



Exercise 3. Show that $\int_1^\infty \frac{1}{x}\,dx$ diverges.

Solution.

$$\begin{aligned}
\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \left[\ln|x|\right]_{x=1}^b \\
&= \lim_{b\to\infty} (\ln|b| - \ln|1|) \\
&= \lim_{b\to\infty} \ln|b| \\
&= \infty.
\end{aligned}$$

□

2 Integrals over unbounded y-values

Example 4.

$$\int_0^1 \ln x\,dx$$

Example 5. (Fig. 1, p. 442 in the textbook)

$$\int_{-2}^1 x^{-2}\,dx$$

Assignment 11: due Friday, April 27

Section 7.4, Problem 29. This will be graded carefully. It will be discussed in tutorial on Thursday.

Section 8.1, Problems 9 and 10. Problem 10 will be graded carefully.

Section 8.2, Problems 3–6, 19, and 21. Problems 4, 6, 19, and 21 will be graded carefully.

Assignment 12: due Monday, April 30

Section 8.2, Problems 7 and 8. Problem 8 will be graded carefully.

Section 8.3, Problems 1–6. Problems 2, 4, and 6 will be graded carefully.

Section 8.4, Problems 1 and 2. Neither of these will be graded carefully, unless your tutor tells you otherwise.

Math 133, Lecture 14: Infinite sequences

Charles Staats

Friday, 30 April 2012

1 What is an infinite sequence?

Definition (1)**.** An *infinite sequence* $\{a_n\}$ is an infinite list of numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots .$$

Example 1. The infinite sequence

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$$

is given by the formula

$$a_n = 1 - \frac{1}{n}.$$

Thus,

$$\begin{aligned}
a_1 &= 1 - \frac{1}{1} = 0 \\
a_2 &= 1 - \frac{1}{2} = \frac{1}{2} \\
a_3 &= 1 - \frac{1}{3} = \frac{2}{3} \\
a_4 &= 1 - \frac{1}{4} = \frac{3}{4} \\
a_5 &= 1 - \frac{1}{5} = \frac{4}{5} \\
a_6 &= 1 - \frac{1}{6} = \frac{5}{6} \\
&\vdots \qquad\quad \vdots \\
a_n &= 1 - \frac{1}{n} = \frac{n-1}{n}.
\end{aligned}$$

Note that there is also a function f defined by the same formula,

$$f(x) = 1 - \frac{1}{x}.$$

This motivates the more formal definition.

Definition (2)**.** An *infinite sequence* $\{a_n\}$ is a function a whose domain consists only of the positive integers. Thus, $a(x)$ is defined when $x = 1, 2, 3, 4, \ldots,$ but not (necessarily) when $x = \frac{3}{2}$ or $x = \pi$. Customarily, we denote $a(n)$ by a_n.

Thus, a sequence is a "rule" that associates to each n a number a_n. Given an infinite list (as in our previous definition), we define a_n to be the n^{th} item on the list.

The "rule" is not always given by an obvious formula.

Example 2 (The Fibonacci Sequence)**.** In the infinite sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

each number is obtained by adding the previous two. We write this "rule" as

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-2} + a_{n-1} \qquad\qquad \text{for } n \geq 3. \end{aligned}$$

Note that a_1 and a_2 have to be defined explicitly because for the first and second item on the list, there are no "previous two items" to add.

2 Limits and of sequences

Intuitively, we say that

$$\lim_{n\to\infty} a_n = \ell,$$

if "$a_\infty = \ell$." However, remember the infinity beast: you want to see what it sees and go where it goes, but you can't touch it directly. So, you use a "saddle," made up of the notions of "arbitrarily [large/small]" and "sufficiently [large/small]," to insulate yourself from actually touching the beast.

$$\underbrace{\text{For arbitrarily small } \varepsilon,}_{\forall\varepsilon>0,}\underbrace{\text{for sufficiently large } n,}_{\exists N \text{ s.t. } \forall n\geq N,}\underbrace{a_n \text{ is within } \varepsilon \text{ of } \ell.}_{|a_n-\ell|<\varepsilon.}$$

It may also help to recall the "courtroom metaphor:" the opponent goes first (choosing ε), then we get to say what "sufficiently large" means (choosing N), and finally the judge decides "who won" by saying whether, for all $n \geq N$, a_n is within ε of ℓ. Giving an ε-N proof amounts to planning out an unbeatable strategy before we even go into the courtroom.

Definition. Each of the statements

- $\lim\limits_{n\to\infty} a_n = \ell$
- $a_n \to \ell$ as $n \to \infty$
- the sequence a_n *converges* to ℓ

means the following:

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geq N, |a_n - \ell| < \varepsilon.$$

Example 3. Show that

$$\frac{1}{n} \to 0$$

as $n \to \infty$.

[Include picture here.]

Proof. Let $\varepsilon > 0$ be given. Set $N = \frac{2}{\varepsilon}$. Then for all $n \geq N$, we have

$$\begin{aligned}
|a_n - \ell| &= \left| \frac{1}{n} - 0 \right| \\
&= \left| \frac{1}{n} \right| \\
&= \frac{1}{n} \\
&\leq \frac{1}{N} && \text{since } n \geq N \\
&= \frac{1}{2/\varepsilon} \\
&= \frac{\varepsilon}{2} \\
&< \varepsilon && \text{since } \varepsilon > 0.
\end{aligned}$$

□

All of the standard limit theorems we learned earlier apply; for instance,

$$\lim_{n\to\infty} (a_n \pm b_n) = \left(\lim_{n\to\infty} a_n \right) \pm \left(\lim_{n\to\infty} b_n \right),$$

provided that the expression on the right makes sense. Unfortunately, you cannot exactly apply l'Hôpital's Rule to a sequence, since Δn can only ever be an integer, so the "derivative"

$$\xcancel{\frac{da}{dn} = \lim_{\Delta n \to 0} \frac{\Delta a_n}{\Delta n}}$$

makes no sense. However, the following more or less obvious theorem often saves us.

Theorem. Let f be a function on the positive real numbers, and define a sequence a_n by

$$a_n = f(n).$$

$$\text{If } \lim_{x\to\infty} f(x) = \ell, \text{ then } \lim_{x\to\infty} a_n = \ell.$$

Proof. The statement $\lim_{x\to\infty} f(x) = \ell$ means precisely that

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall x \geq N, |f(x) - \ell| < \varepsilon.$$

Since $a_n = f(n)$, this implies that

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geq N, |a_n - \ell| < \varepsilon,$$

which is precisely what it means to say that $\lim_{x\to\infty} a_n = \ell$. □

Exercise 4. Show that $a_n = (\ln n)/e^n$ converges to 0.

Solution.

□

Sometimes, we can determine *whether* a sequence *converges* or *diverges*[1], even if we cannot compute its limit.

Example 5. Define sequences a_n and b_n by

$$a_n = \sum_{i=1}^{n} \frac{1}{n^2}$$
$$b_n = \sum_{i=1}^{n} \frac{1}{n^3}.$$

Using the techniques of the next two sections, it will be easy to see that both a_n and b_n converge. However, it is highly non-obvious that

$$a_n \to \frac{\pi^2}{6},$$

and there is *no* known formula for the limit of b_n other than "the limit of this sequence."

[Discuss axiom: if a sequence is nondecreasing and bounded above, then it converges.]

[1] A sequence *diverges* if it does not converge.

Assignment 13: due Wednesday, May 2

Section 7.4, Problems 21–23. Problems 22 and 23 will be graded carefully.

Section 8.3, Problems 7, 8, and 25. (On 25, use trig substitution rather than "partial fractions.") Problems 8 and 25 will be graded carefully. Problem 25 will be discussed in tutorial on Tuesday.

Section 8.4, Problems 3–6. Problems 4 and 6 will be graded carefully.

Assignment 14: due Friday, May 4

Section 8.2, Problems 9 and 10. Problem 10 will be graded carefully.

Section 8.4, Problems 9 and 10. Problem 10 will be graded carefully.

Section 9.1, Problems 1–4. Problems 2 and 4 will be graded carefully.

Consider the two functions

$$\begin{aligned} f(x) &= 1 \\ g(x) &= \cos^2(\pi x). \end{aligned}$$

(a) Graph these two functions for $0 \leq x \leq 8$.

(b) Show that the two sequences

$$\begin{aligned} a_n &= 1 \\ b_n &= \cos^2(\pi n) \end{aligned}$$

are in fact the same sequence.

(c) Explain why this seems to contradict the second definition of *infinite sequence* in Lecture 14, but does not actually contradict the definition.

This problem will be discussed in tutorial on Thursday. It will be graded carefully.

Math 133, Lecture 15: Infinite series

Charles Staats

Wednesday, 2 May 2012

1 Review of ε-N definition of convergence

Let $\{a_n\}_{n=1}^{\infty}$ be an infinite sequence. The *tails* of the sequence are the "subsequences" obtained by removing the first $N-1$ terms, for some N. For instance, the tails for $N = 5$ and $N = 17$ are the sequences

$$\begin{aligned} a_5,\, a_6,\, a_7,\, a_8,\, a_9, \ldots \qquad & (N = 5) \\ a_{17},\, a_{18},\, a_{19},\, a_{20},\, a_{21}, \ldots \qquad & (N = 17) \end{aligned}$$

Example 1. Let $a_n = \frac{1}{n}$. Then the tails for $N = 5$ and $N = 17$ are

$$\begin{aligned} \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \ldots \qquad & (N = 5) \\ \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}, \frac{1}{21}, \ldots \qquad & (N = 17) \end{aligned}$$

We say that the sequence $\{a_n\}_{n=1}^{\infty}$ is *ultimately within* the interval $(\ell-\varepsilon, \ell+\varepsilon)$ if one of its tails is within this interval. For instance, in the sequence $a_n = \frac{1}{n}$, the sequence is ultimately within the interval $(-\frac{1}{16}, \frac{1}{16})$ because its seventeenth tail lies within this interval:

[picture here]

This is true even though neither the sequence nor its fifth tail lie within the interval. (Both the sequence and its fifth tail contain the term $\frac{1}{16}$, which does not lie within the interval $(-\frac{1}{16}, \frac{1}{16})$.)

The sequence $\{a_n\}$ *converges* to ℓ if for every $\varepsilon > 0$, the sequence is ultimately within the interval $(\ell - \varepsilon, \ell + \varepsilon)$.

It should strike you as rather remarkable that we are able to prove any sequence converges at all, using this definition. For instance, for the sequence $a_n = \frac{1}{n}$, if I give you an $\varepsilon > 0$, it will not be too difficult for you to find a tail that lies within the interval $(-\varepsilon, \varepsilon)$. But given a finite amount of time to work, how do you find tails for *every single* ε?

2 Terminology: sequence versus series

Last lecture we dealt with infinite *sequences*, i.e., infinite lists:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

This lecture we will deal with infinite *series*, i.e., infinite *sums*:

$$\sum_{n=1}^{\infty} a_n.$$

The terminology here could be confusing, but is important to remember:

- An infinite *sequence* is an infinite *list*.
- An infinite *series* is an infinite *sum*.

2.1 Partial Sums

Given any series

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_i + \cdots ,$$

there is an associated *sequence of partial sums*, defined by

$$S_n = \sum_{i=1}^{n} a_i.$$

Thus,

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n \\
&\vdots
\end{aligned}$$

Exercise 2. Consider the series

$$\sum_{i=1}^{\infty} 10^{-i} = 0.1 + 0.01 + 0.001 + 0.0001 + \cdots .$$

What is its sequence of partial sums?

Solution.

$$0.1,\ 0.11,\ 0.111,\ 0.1111,\ \ldots .$$

□

3 Convergence

A series is said to *converge* if its sequence of partial sums converges (to a finite limit); it *diverges* if the sequence of partial sums diverges. If the series converges, we say that the infinite sum is the limit of the partial sums.

Notationally,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

is how we *define* the left-hand side.

Sometimes, we can determine *whether* a series *converges* or *diverges*, even if we cannot compute its limit.

Theorem (Integral test for positive series)**.** If f is a continuous, nonincreasing, positive function on $[1, \infty)$, then the series $\sum_{i=1}^{\infty} f(i)$ and the improper integral $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

It is important to note that the integral test can tell us *whether* a series converges, but not what it converges *to*.

Example 3. Consider the series

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

$$\sum_{i=1}^{\infty} \frac{1}{i^3}.$$

The integral test shows that both of these series converge:

However, it is not at all obvious that

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6},$$

and there is *no* known formula for the limit of $\sum_{i=1}^{\infty} \frac{1}{i^3}$ better than $\sum_{i=1}^{\infty} \frac{1}{i^3}$.

Assignment 14: due Friday, May 4

Section 8.2, Problems 9 and 10. Problem 10 will be graded carefully.

Section 8.4, Problems 9 and 10. Problem 10 will be graded carefully.

Section 9.1, Problems 1–4. Problems 2 and 4 will be graded carefully.

Consider the two functions

$$f(x) = 1$$
$$g(x) = \cos^2(\pi x).$$

(a) Graph these two functions for $0 \leq x \leq 8$.

(b) Show that the two sequences

$$a_n = 1$$
$$b_n = \cos^2(\pi n)$$

are in fact the same sequence.

(c) Explain why this seems to contradict the second definition of *infinite sequence* in Lecture 14, but does not actually contradict the definition.

This problem will be discussed in tutorial on Thursday. It will be graded carefully.

Assignment 15: due Monday, May 7

Section 8.3, Problems 11 and 12. Problem 12 will be graded carefully.

Section 9.1, Problems 5, 6, and 21–24. Problems 6, 22, and 24 will be graded carefully.

Section 9.2, p. 460, Concepts Review 1–4. Try to do these without looking anything up. Then check your answers.

Section 9.3, Problem 1.

Math 133, Lecture 16: Infinite series, continued

Charles Staats

Friday, 4 May 2012

Quiz retake policy

Given that quizzes are more important in this class as a learning tool than as a testing tool, I've decided on the following policy for retaking quizzes: If you took a quiz, and are not satisfied with your performance, you may contact me to schedule a re-take at a mutually convenient time. You may re-take several different quizzes at once. However, the following limitations apply:

1. If you missed a quiz because of an un-excused absence from tutorial, your grade for that quiz will go down as a zero, regardless of any re-takes. (However, if you just want to re-take the quiz for practice, I will be willing to give you an "unofficial" grade on a re-take.)

2. Once a test has been given, you may not officially re-take any quizzes from before the test. (This is to discourage last-minute studying for the final exam.)

I may at some point give a survey to find out how helpful people find the "easy" quizzes versus the "test-like" quizzes. In the mean time, please feel free to ask me questions of the following type:

1. I have just written an answer to this question (from the homework, or a list of possible test questions, or ...). Would you grade it to give me an idea how I would have done if I wrote this on the test?

2. Would you please write up a solution to such-and-such a question?

I will not necessarily answer "yes" to either of these, but there is no harm in asking. I would also note that a request for a solution writeup will be more persuasive if it comes from someone who has already done their own writeup, or if more than one person asks for a writeup of the same question.

1 Some basic facts about series convergence

Axiom (Monotone Convergence)**.** A positive series either converges, or diverges to infinity.

This is called a "theorem" by the book, but it's really more of an axiom. If you want to discuss this in greater detail, come to my office hours or make an appointment.

Note how this was used in proving the Integral Test: We showed, via a picture, that (for f a positive decreasing function)

$$\sum_{i=1}^{\infty} f(i) \leq f(1) + \int_1^{\infty} f(x)\, dx.$$

Thus, if the right-hand side is finite, then the series on the left-hand side cannot diverge to infinity. Consequently, it must converge.

You may be asked on a test to explain how the Monotone Convergence Axiom is used in the proof of the Integral Test.

Here's another basic result.

Theorem. If $\displaystyle\sum_{i=1}^{\infty} a_i$ converges, then $a_i \to 0$ as $i \to \infty$.

Proof. The statement should strike you as more or less "obvious," but here is a proof just in case it does not. Assume

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

converges. Then

$$\begin{aligned}
a_n &= \left(\sum_{i=1}^{n} a_i\right) - \left(\sum_{i=1}^{n-1} a_i\right) \\
\lim_{n\to\infty} a_n &= \left(\lim_{n\to\infty} \sum_{i=1}^{n} a_i\right) - \left(\lim_{n\to\infty} \sum_{i=1}^{\boxed{n-1}} a_i\right) \\
&= \left(\sum_{i=1}^{\infty} a_i\right) - \left(\sum_{i=1}^{\infty} a_i\right) \\
&= 0 && \text{since } \textstyle\sum_{i=1}^{\infty} a_i \text{ is finite.}
\end{aligned}$$

□

Exercise 1. What is the contrapositive of the statement of this theorem? Can you use the theorem to prove that a series converges?

Solution. $\square$

There are also some basic properties of sums that extend to infinite sums in a natural manner.

Theorem. 1. (Distributive Law) If c is a nonzero constant, then

$$\sum_{i=1}^{\infty} ca_i = c\sum_{i=1}^{\infty} a_i.$$

In particular, the sum on the right-hand side converges *if and only if* the sum on the left-hand side converges.

2.

$$\sum_{i=1}^{\infty}(a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i,$$

provided that the two series on the right-hand side converge.

Exercise 2. Give an example of two series $\sum a_i$, $\sum b_i$, neither of which converges, but such that $\sum a_i + b_i$ does converge. (Hint: try $a_i = 1$ and $b_i = -1$.)

One final note is that whether or not a series converges depends only on its tail.

2 Some standard series

There are a few series you should simply know.

1. A *geometric series* is a series of the form

$$\sum_{i=0}^{\infty} ar^i = a + ar + ar^2 + ar^3 + \cdots,$$

where $a \neq 0$. If $|r| < 1$, the series converges to

$$\frac{a}{1-r};$$

if $|r| \geq 1$, the series diverges. Whether the series converges or diverges can be proved using the integral test; for a fairly nice explanation of what it converges *to*, see Example 1, p. 456 in the textbook. I'll try to give a geometric explanation in class if I have time.

2. A p-series is a series of the form

$$\sum_{i=1}^{\infty} \frac{1}{i^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots.$$

The series converges if and only if $p > 1$. However, as I mentioned last lecture, it is not at all easy to say what the series converges *to*.

Example 3. A coin is tossed as many times as necessary until it lands on "heads." What is the probability that the coin does, in fact, eventually land on "heads"?

Solution.

□

3 The ratio test

Consider the series

$$\sum_{k=0}^{\infty} \frac{2^k}{k!}.$$

We will see later that this series is extremely important, and converges to e^2. The first step in doing this is to prove that this series, and other series like it, converge. However, the integral test is not feasible, because $k!$ is only defined when k is an integer.[1]

In situations like this, we need other tests for convergence. And, as it turns out, these tests are often easier to use than the integral test. For today, we'll concentrate on the Ratio Test.

Theorem (Ratio Test)**.** If $\sum a_n$ is a positive series (i.e., all the a_n are positive) and

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho$$

exists, we have:

(i) If $\rho < 1$, the series converges.

(ii) If $\rho = 1$, anything could happen. (The Ratio Test is inconclusive, and we need to try something else.)

(iii) If $\rho > 1$ (in particular, if $\rho = \infty$), then the series diverges.

Idea: the series behaves like a geometric series.

Example 4 (Textbook, p. 472, Example 5)**.** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

converges or diverges.

[1] It is possible to extend $k!$ to a natural function on $[1, \infty)$, but the function that results is not elementary or easy to integrate. In any case, even making this definition requires the tools we are now developing.

Assignment 15: due Monday, May 7

Section 8.3, Problems 11 and 12. Problem 12 will be graded carefully.

Section 9.2, p. 460, Concepts Review 1–4. Try to do these without looking anything up. Then check your answers.

Section 9.3, Problem 1.

Assignment 16: due Wednesday, May 9

Section 8.4, Problems 11 and 12. Problem 12 will be graded carefully.

Section 9.1, Problems 25 and 26. Problem 26 will be graded carefully.

Section 9.2, Problem 33. This will be discussed in tutorial on Tuesday. It will be graded carefully.

Section 9.3, Problems 2–5. Problems 2 and 4 will be graded carefully.

Section 9.4, Problems 5 and 6. Problem 6 will be graded carefully.

Math 133, Lecture 17: Infinite series, continued

Charles Staats

Monday, 7 May 2012

Test Next Week

I anticipate giving a test next week (week 8), probably on Wednesday.

1 Improved explanation from last time

Last lecture, I gave a rather incoherent explanation of the fact that, if the series $\sum_n a_n$ converges, then $a_n \to 0$. Here is a (hopefully) more coherent explanation of that fact.

If $\sum_n a_n$ converges to s, then for every $\varepsilon > 0$, the partial sums are "ultimately within ε of s." Consequently, the sequence a_n must be "ultimately within ε of 0;" otherwise, if some term a_n were *not* within ε of 0, adding a_n would kick the partial sum outside ε of s.

[picture]

2 Two comparison tests

Theorem (Ordinary Comparison Test)**.** Suppose $\sum_n a_n$ and $\sum_n b_n$ are positive series such that $a_n \leq b_n$ for all n. If $\sum_n b_n$ converges, then $\sum_n a_n$ converges too.

Proof. This proof is not precisely rigorous—we should really be working with partial sums—but I think you will be convinced. If $a_n \leq b_n$ for all n, then

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n < \infty,$$

where we know $\sum_n b_n < \infty$ since $\sum_n b_n$ converges. Since $\sum_{n=1}^{\infty} a_n$ is a positive series that does not diverge to infinity, it converges. □

Essentially, we're saying that "if $a_n \leq b_n$, and $\sum_n b_n$ does not diverge to infinity, then neither does $\sum_n a_n$.

Here's another test along the same lines, but a bit easier to use (most of the time) and a bit harder to prove (see p. 470 of the textbook).

Theorem (Limit Comparison Test)**.** Suppose that $\sum_n a_n$ and $\sum_n b_n$ are positive series (i.e., the a_n and b_n are all positive)[1]. Also assume that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists.

- If $0 < \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} < \infty$, then $\sum_n a_n$ and $\sum_n b_n$ converge or diverge together.
- If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then if $\sum_n b_n$ converges, then $\sum_n a_n$ also converges.

The idea of the proof is the following: To determine the convergence or divergence of a series, we can look only at the "tail," i.e., ignore the first N terms, for any finite N. Once we do this, the limit will tell us that the series ultimately behave like constant multiples of each other.

Example 1. (Example 4, p. 471 in the textbook) Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.

Exercise 2. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges or diverges.

Solution.

□

[1]This also works if the a_n are allowed to be zero, as long as they don't turn negative.

3 Ratio test, recalled

Let us recall the ratio test, which allows us to "compare a series to itself" to determine whether it converges or diverges—at least sometimes.

Theorem (Ratio Test)**.** If $\sum a_n$ is a positive series (i.e., all the a_n are positive) and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

exists, we have:

(i) If $\rho < 1$, the series converges.

(ii) If $\rho = 1$, anything could happen. (The Ratio Test is inconclusive, and we need to try something else.)

(iii) If $\rho > 1$ (in particular, if $\rho = \infty$), then the series diverges.

Idea: the series behaves like a geometric series.

Example 3 (Textbook, p. 472, Example 5)**.** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

converges or diverges.

4 Absolute convergence

So far, we have concerned ourselves almost exclusively with "positive series." Here's a theorem showing how to apply these same techniques to series with negative terms:

Definition. We say the series $\sum_n a_n$ *converges absolutely* if $\sum_n |a_n|$ converges.

Theorem. If a series converges absolutely, then it converges.

In fact, if a series converges, but does not converge absolutely, it is not a very nice kind of convergence, for the following (somewhat surprising reason): If a series converges, but not absolutely, then you can change what it converges to by re-arranging the order of the terms. Thus, such a series is called *conditionally convergent.*

We will be concerned primarily with whether a series converges absolutely. It is a less surprising, but difficult to prove, theorem that if a series converges absolutely, then rearranging the order of its terms does not affect what it converges to.

Assignment 16: due Wednesday, May 9

Section 8.4, Problems 11 and 12. Problem 12 will be graded carefully.

Section 9.1, Problems 25 and 26. Problem 26 will be graded carefully.

Section 9.2, Problem 33. This will be discussed in tutorial on Tuesday. It will be graded carefully.

Section 9.3, Problems 2–5. Problems 2 and 4 will be graded carefully.

Section 9.4, Problems 5 and 6. Problem 6 will be graded carefully.

Assignment 17: due Friday, May 11

Find all possible solutions to the differential equation

$$\frac{d^3y}{dx^3} = 1.$$

(Hint: antidifferentiate three times.) Your "solution family" should have three different constants. Find formulas for $y(0)$, $y'(0)$, and $y''(0)$ in terms of these three constants. What should the value of the constants be to ensure that $y(0) = 1$, $y'(0) = 1$, and $y''(0) = 1$?

This problem will be discussed in tutorial on Thursday. It will be graded carefully.

Section 9.3, Problems 6–8 and 13–14. In 13–14, you may use any test we have discussed, even if it comes after Section 9.3. Problems 6, 8, and 14 will be graded carefully.

Section 9.4, Problems 1, 2, 11, and 12. Problems 2 and 12 will be graded carefully.

Bonus problem: Section 9.5, Problem 35.

Math 133, Lecture 19: Power series, continued

Charles Staats

Friday, 11 May 2012

Test Wednesday

This section is just to note that there will be a test Wednesday.

1 Absolute Ratio Test

We covered the ratio test for convergence of positive series, but I did not cover the Absolute Ratio Test very well. Since it is important to computing convergence intervals for power series, I thought I should probably state it carefully.

Theorem (Absolute Ratio Test)**.** Let $\sum_n u_n$ be an infinite series, *not necessarily positive*, such that for all n, $u_n \neq 0$. Suppose that the limit

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \rho$$

exists. Then we have

(i) If $\rho < 1$, the series converges *absolutely* (and, in particular, converges).

By definition, $\sum_n u_n$ converges *absolutely* if the positive series $\sum_n |u_n|$ converges. By the Ratio Test we saw before, $\sum_n |u_n|$ converges since $|u_{n+1}|/|u_n| \to \rho < 1$.

(ii) If $\rho = 1$, anything could happen.

(iii) If $\rho > 1$, the series $\sum_n u_n$ *diverges.*

The proof is essentially this: if $\lim_{n\to\infty} |u_{n+1}|/|u_n| = \rho$, then $|u_n|$ eventually looks a lot like

$$a, \rho a, \rho^2 a, \rho^3 a, \ldots$$

for some $a > 0$. Since $\rho > 1$, we see that $|u_n| \to \infty$. But for the series $\sum_n u_n$ to converge, we would have to have $u_n \to 0$.

2 The radius of convergence of a power series

Remember that a power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n.$$

Significantly, this is a series of *functions*, rather than a series of numbers. Thus, we should not ask simply *whether* it converges, but rather, *for which values of* x (i.e., *where*) does it converge?

A basic example to start with is the series

$$\sum_{n=0}^{\infty} x^n,$$

in which the coefficient c_n is equal to 1 for every n. We saw, previously, that this is a *geometric series*: it converges (absolutely) to

$$\frac{1}{1-x}$$

when $|x| < 1$, and diverges when $|x| > 1$.

The following theorem shows that this is typical convergence behavior for power series:

Theorem. Let $\sum_n c_n x^n$ be a power series. Then there is a nonnegative "number" R (which could be in the interval $[0, \infty)$, or could in fact be ∞) such that

(i) the series converges absolutely on the interval $(-R, R)$, and

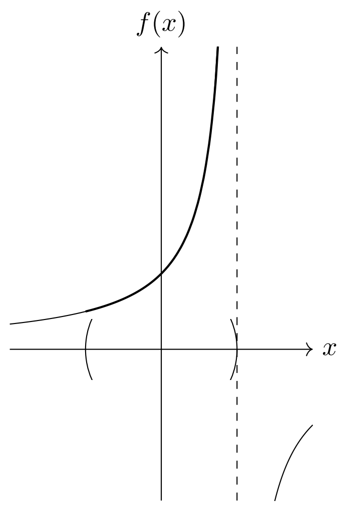
(ii) the series diverges when $|x| > R$.

Note that this theorem says absolutely nothing about what happens when $x = R$; this should remind you of the Ratio Test and the Absolute Ratio Test.

If you think about it, this theorem is really quite remarkable. In the example we've looked at, we have that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} =: f(x)$$

where it converges. If you look at the graph of f,



it seems quite reasonable that the series should be forced to diverge at $x = 1$, since the function itself goes to infinity. But the function is perfectly smooth at $x = -1$, so how does the series "know" it needs to stop converging for $x < -1$?

A "proper" proof of this theorem should really use some analysis that the book skirts over. I would love to take the time to explain this properly, and in the process give you a better feel for exactly what it means that the "real numbers have no holes" and why you should believe the Monotone Convergence Axiom (a positive series either converges, or diverges to infinity). Unfortunately, I don't have the luxury of unlimited time. So instead, I will offer you a "plausibility argument" that resembles how the Ratio of Convergence is actually computed in some examples, and invite anyone who wants a better understanding to come to my office hours or schedule an appointment.

Plausibility argument. Suppose we have a power series

$$\sum_{n=0}^{\infty} c_n x^n$$

with all the coefficients c_n nonzero, and such that the limit

$$T = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$$

exists. Let's see what the Absolute Ratio Test tells us.

$$\begin{aligned}\lim_{n\to\infty} \frac{|c_{n+1}x^{n+1}|}{|c_n x^n|} &= \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} \cdot |x| \\ &= |x| \cdot \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} \\ &= |x| \cdot T.\end{aligned}$$

Thus,

- For every x-value such that $|x| < \dfrac{1}{T}$, we have

$$\begin{aligned} |x| &< \frac{1}{T} \\ |x| \cdot T &< 1 \\ \lim_{n\to\infty} \frac{|c_{n+1}x^{n+1}|}{|c_n x^n|} &< 1; \end{aligned}$$

  hence, by the Absolute Ratio Test, $\sum_n c_n x^n$ converges absolutely when $|x| < \dfrac{1}{T}$.

- When $|x| > \dfrac{1}{T}$, we have

$$\lim_{n\to\infty} \frac{|c_{n+1}x^{n+1}|}{|c_n x^n|} = |x| \cdot T > \frac{1}{T} \cdot T = 1,$$

  hence $\sum_n c_n x^n$ diverges by the Absolute Ratio Test.

Thus, $\sum_n c_n x^n$ converges absolutely when $|x| < 1/T$, and diverges when $|x| > 1/T$. In other words, the theorem holds, with radius of convergence

$$R = \frac{1}{T}. \qquad \square$$

3 Power series in $x - a$

If we have a series of the form

$$\sum_n c_n (x-a)^n,$$

substitute

$$u = x - a,$$

and consider this as a power series in u. Then do all the computations, and finally reverse the substitution.

Example 1 (Example 5, p. 483)**.** Find the convergence set for $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+1)^2}$. Ignore the endpoints.

Assignment 18: due Monday, May 14

Section 9.4: Concepts Review 1–4; Problems 13–16. Problems 14 and 16 will be graded carefully.

Section 9.6, Problems 9, 10, and 29. Problem 29 will be graded carefully.

Test Wednesday, May 16

The test will cover lectures 1–18, with emphasis on lectures 8–18. The assignments corresponding to the emphasized material are Assignments 8–18. I should note that some problems will require techniques from earlier.

Assignment 19: due Friday, May 18

Section 9.6, Problems 11–14. You don't need to say what happens at the endpoints. Problems 12 and 14 will be graded carefully.

Math 133, Lecture 20: Taylor and Maclaurin Series

Charles Staats

Monday, 14 May 2012

Test Wednesday

This section is just to note that there will be a test Wednesday.

1 A common error

There is an error that many people made on a quiz that I have been meaning to bring up for some time, but have kept forgetting. Perhaps I will remember if I include it in my lecture notes.

Remember that for any *series*

$$\sum_{i=1}^{\infty} a_i$$

there is a corresponding *sequence*, the "sequence of partial sums," given by

$$S_n = \sum_{i=1}^{n} a_i.$$

Thus, in "expanded version," the *series* is the sum

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

while the *sequence* of partial sums is the list

$$\underbrace{a_1}_{S_1}, \quad \underbrace{a_1 + a_2}_{S_2}, \quad \underbrace{a_1 + a_2 + a_3}_{S_3}, \quad \underbrace{a_1 + a_2 + a_3 + a_4}_{S_4}, \ldots .$$

The sequence of partial sums is significant in that the sum of the series is defined to be the limit of the sequence of partial sums:

$$\sum_{i=1}^{\infty} a_i := \lim_{n\to\infty} S_n.$$

2 Uniqueness of power series expansion

We have seen that, sometimes, a function $f(x)$ may be represented by a power series that converges absolutely to f on an interval. For instance, we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

where the first equality holds on the interval $(-1, 1)$ and the second holds on the interval $(-\infty, \infty)$. The first equality was computed via a geometric series; the second was an apparently miraculous result of the fact that the power series $\sum_n x^n/n!$ is its own derivative.

A number of natural questions occur at once here. Two of them are these: First, is there some method (other than a "miraculous guess") that would allow us to obtain the power series for e^x? Second, if we produce power series for the same function by two different methods[1], are they guaranteed to be the same? In both cases, the answer is "yes." When we know a function, we will give a formula for the only possible power series that might converge to it. This power series series is not *guaranteed* to converge to the function; but it is the only power series that possibly could, and we have a formula for it.

Remember the following theorem: if the power series $\sum_n c_n x^n$ converges to the function $f(x)$ on an open interval, then the term-by-term derivative of the power series converges to $f'(x)$ on the same open interval. Thus, suppose we have $f(x) = \sum_{n=0}^{\infty} c_n x^n$ on some open interval. Then on that interval, we will have

$$\begin{aligned}
f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots \\
f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + (n+1)c_{n+1} x^n + \cdots \\
f''(x) &= 2c_2 + (3\cdot 2)c_3 x + (4\cdot 3)c_4 x^2 + (5\cdot 4)c_5 x^3 + \cdots + (n+2)(n+1)c_{n+2} x^n + \cdots \\
f'''(x) &= 3!c_3 + (4\cdot 3\cdot 2)c_4 x + (5\cdot 4\cdot 3)c_5 x^2 + (6\cdot 5\cdot 4)c_6 x^3 + \cdots + (n+3)(n+2)(n+1)c_{n+3} x^n + \cdots
\end{aligned}$$

[1]This question will seem more interesting once we have seen different methods to produce power series.

Evaluating at $x = 0$, all terms except the constant term disappear:

$$\begin{aligned} f(0) &= c_0 \\ f'(0) &= c_1 \\ f''(0) &= 2c_2 \\ f'''(0) &= 3!c_3 \\ f^{(4)}(0) &= 4!c_4 \\ &\vdots \\ f^{(n)}(0) &= n!c_n \end{aligned}$$

Solving the general equation for c_n, we find that

$$c_n = \frac{f^{(n)}(0)}{n!}.$$

(Note that $f^{(0)}$ is obtained by "differentiating f zero times," i.e., doing nothing at all to f; consequently, $f^{(0)}(x) = f(x)$. Similarly, $0! = 1$.) This shows the following theorem:

Theorem. If

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where the series converges on an open interval including the point $x = 0$, then f has all of its derivatives defined at 0, and

$$c_n = \frac{f^{(n)}(0)}{n!}.$$

Thus, given a function (with infinitely many derivatives at 0), we can produce a power series, called the *Maclaurin series*, for the function, by the formula

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

In particular, if the function in question is $f(x) = e^x$, then we have

$$f^{(n)}(x) = f^{(n-1)}(x) = \cdots = f'(x) = f(x) = e^x;$$

thus,

$$f^{(n)}(0) = e^0 = 1$$

for all n, and so the Maclaurin series for e^x is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Note that, if someone gives us a function f and we compute its Maclaurin series, it is *not* guaranteed that the Maclaurin series converges to f. Rather, the Maclaurin series is the "only hope" for a power series that converges to f.

Test Wednesday, May 16

The test will cover lectures 1–18, with emphasis on lectures 8–18. The assignments corresponding to the emphasized material are Assignments 8–18. I should note that some problems will require techniques from earlier.

Assignment 19: due Friday, May 18

Section 9.6, Problems 11–14. You don't need to say what happens at the endpoints. Problems 12 and 14 will be graded carefully.

Math 133, Lecture 21: Taylor Series

Charles Staats

Friday, 18 May 2012

1 Maclaurin series of polynomials

Last lecture, as you hopefully recall, we made the following definition:

Definition. The *Maclaurin series* for a function f is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

where $f^{(n)}(x)$ denotes the n^{th} derivative of f at x and, by convention the "zeroth derivative" is just f itself.

This definition came out of a computation, which also proved the following theorem:

Theorem. If f is a function such that *some* power series

$$\sum_{n=0}^{\infty} c_n x^n$$

converges to f with a positive radius of convergence, then this power series is *necessarily* the Maclaurin series for f. In other words, f has infinitely many derivatives at 0, and

$$c_n = \frac{f^{(n)}(0)}{n!}$$

for all n.

One can get a "cute" application by looking at *finite* power series, i.e., polynomials.

Example 1. Let f be the polynomial

$$f(x) = x^3 - 7x^2 + 2x + 9.$$

Give the formula, in "standard form," for the translation of f one unit to the right.

Solution. Let g be the translation of f one unit to the right; in other words,

$$\begin{aligned} g(x) &= f(x-1) \\ &= (x-1)^3 - 7(x-1)^2 + 2(x-1) + 9. \end{aligned}$$

We could figure out the "standard form" for g by expanding the products and combining like terms. But Maclaurin series give us an alternative. Since the "standard form" for g is necessarily going to look like

$$g(x) = \sum_{n=0}^{\infty} c_n x^n$$

(with $c_n = 0$ whenever n is large), the Maclaurin Series theorem tells us that for all n,

$$c_n = \frac{g^{(n)}(0)}{n!}.$$

Thus, let's first differentiate g (without expanding it):

$$\begin{aligned} g(x) &= (x-1)^3 - 7(x-1)^2 + 2(x-1) + 9 \\ g'(x) &= 3(x-1)^2 - 14(x-1) + 2 \\ g''(x) &= 6(x-1) - 14 \\ g'''(x) &= 6 \\ g^{(n)}(x) &= 0 && \text{whenever } n \geq 4. \end{aligned}$$

Thus, we have

$$\begin{aligned} c_0 &= \frac{g(0)}{0!} = \frac{(-1)^3 - 7(-1)^2 + 2(-1) + 9}{1} = -1 - 7 - 2 + 9 = -1 \\ c_1 &= \frac{g'(0)}{1!} = \frac{3(-1)^2 - 14(-1) + 2}{1} = 3 + 14 + 2 &&= 19 \\ c_2 &= \frac{g''(0)}{2!} = \frac{6(-1) - 14}{2} = \frac{-20}{2} &&= -10 \\ c_3 &= \frac{g'''(0)}{3!} = \frac{6}{6} &&= 1 \\ c_n &= 0 \qquad \text{whenever } n \geq 4. \end{aligned}$$

Consequently, we have

$$g(x) = (-1) + 19x - 10x^2 + x^3. \qquad \square$$

Exercise 2. Find the "standard form" of the polynomial

$$g(x) = 2(x+1)^3 + (x+1)^2 - 2(x+1) + 1$$

using Maclaurin series.

Solution. ∎

2 Taylor series

In a previous lecture, I mentioned the possibility of power series "about a," i.e., power series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n.$$

The idea was to substitute in $u = x - a$, do everything for power series in u, and then, at the end, rewrite everything back in terms of x.

Let's see what happens when we do this for the Maclaurin series formula.

Suppose we want to find a power series for f about a, i.e., write

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

Substituting $u = x - a$ and, consequently, $x = u + a$, we obtain

$$\begin{aligned} f(u+a) &= \sum_{n=0}^{\infty} c_n u^n \\ g(u) &= \sum_{n=0}^{\infty} c_n u^n \end{aligned}$$

where $\begin{aligned} g(u) &= f(u+a) \\ g^{(n)}(u) &= f^{(n)}(u+a) \end{aligned}$, the derivatives coming from the Chain Rule. By the Maclaurin Series Theorem,

$$\begin{aligned} g(u) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} u^n \\ f(u+a) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0+a)}{n!} u^n \\ f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n. \end{aligned}$$

Thus, we obtain the following more general formula:

Definition. The *Taylor series about a* for a function f is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Thus, the "Maclaurin series" is identical to the "Taylor series about 0." Thus, there is no real need to remember the term "Maclaurin series."

As in the case of Maclaurin series, we have the following theorem:

Theorem. If we have a function f and a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

that converges to f on an open interval about $x = a$, then this series can only be the Taylor series for f about a.

Exercise 3. Rewrite the polynomial

$$f(x) = 2x - 2x^2 + x^3$$

as a polynomial in $(x - 1)$, by finding its Taylor series about $a = 1$.

Solution.

$\square$

Assignment 20: due Monday, May 21

Section 9.6, Problems 1–4. Ignore the endpoints. Problems 2 and 4 will be graded carefully.

Section 9.8, Problems 1, 2, 19, 20, 23, and 24. Problems 2, 20, and 24 will be graded carefully.

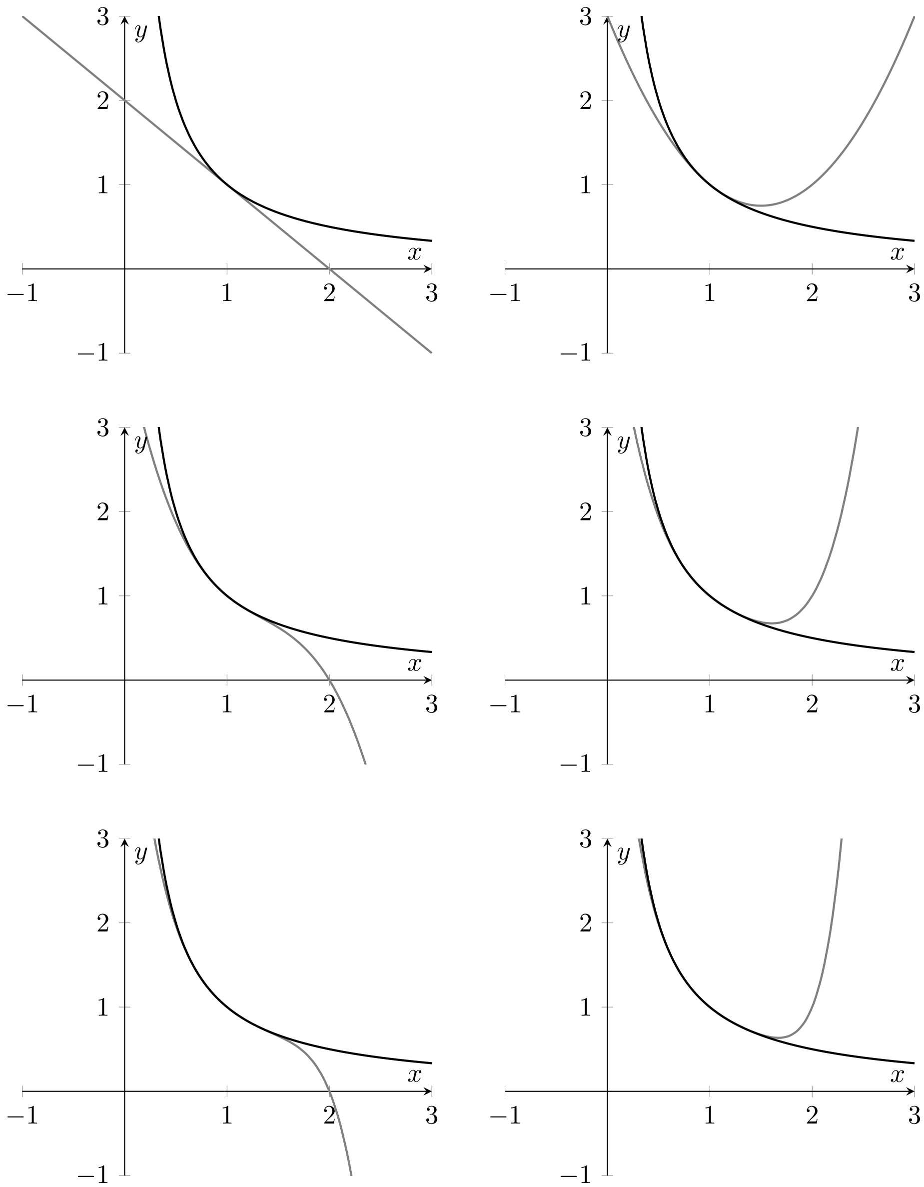
Assignment 21: due Wednesday, May 23

To be decided.

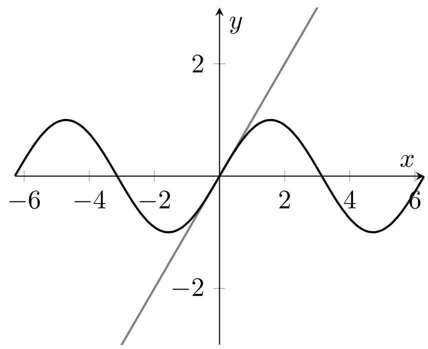
Math 133, Lecture 22: Taylor Series, continued

Charles Staats

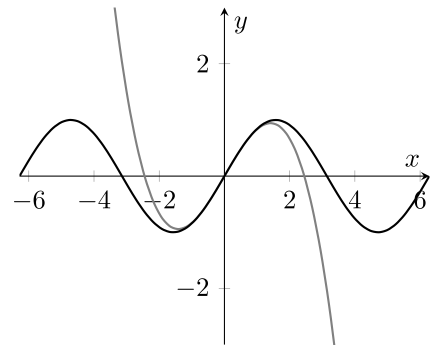
Monday, 21 May 2012



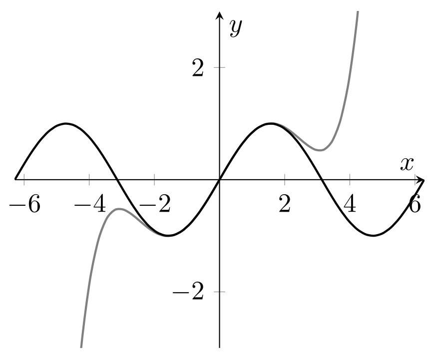
$$y = x$$



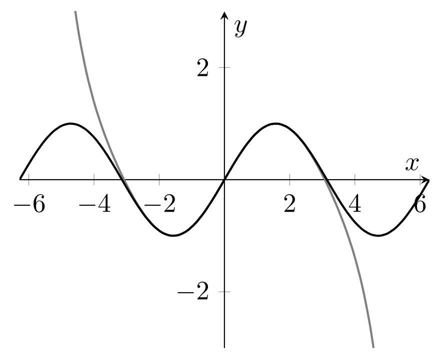
$$y = x - \frac{x^3}{3!}$$



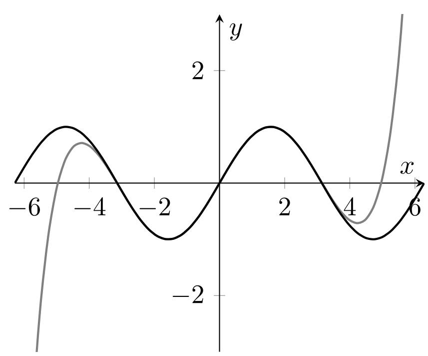
$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$



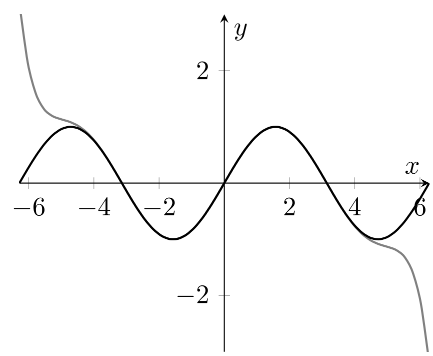
$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$



$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$$



Assignment 21: due Wednesday, May 23

Section 9.7, Problems 1–6. Problems 2, 4, and 6 will be graded carefully.

Assignment 23: due Friday, May 25

To be decided.

Math 133, Lecture 23: Operations on power series

Charles Staats

Wednesday, 23 May 2012

1 Some building blocks

Recall an important theme of Taylor series:

(i) The Taylor series is the only power series that could possibly converge to a function. Consequently,

(ii) If we somehow find a power series that converges to a function, it must be the Taylor series in disguise.

When we know a few basic Taylor series, there are any number of "tricks" that can be used to combine these to give us other Taylor series. Recall the basic "building blocks" we have already found:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \sum_{n=0}^{\infty} x^n \tag{1}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{3}$$

Power series (1) is an example of exactly the sort of thing I am talking about. We know it converges to the function $1/(1-x)$ on the interval $(-1, 1)$ based on our prior understanding of *geometric* series. While we could derivatives to verify that it is, in fact, the Taylor series for $1/(1-x)$, we don't have to: any power series that converges to $1/(1-x)$ is necessarily its Taylor series, no matter how we found it.

The series (2) and (3) were actually computed as Taylor series. I should probably note that we have not, technically, proved—yet—that series (3) actually converges to $\sin x$, only that it is the Taylor series for $\sin x$ and converges to *something*. For the moment, I shall conveniently ignore this gap.

2 Substitution

Let's proceed by examples.

Example 1. Find the Taylor series for $\dfrac{1}{1+x}$ about $x = 0$.

Solution. (Hint: substitute $-x$ for x in a series we already know.)

$\square$

Example 2. Find the Taylor series for e^{-x^2}.

Solution.

$\square$

The examples given so far suggest that substitution is a very "easy" technique—which it often is. But not always.

Example 3. Find the Taylor series for $\sin(\sin x)$.

The trick here would be to compose the Taylor series:

$$\begin{aligned}
\sin(\sin x) &= \sin\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&\quad - \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^3}{3!} \\
&\quad + \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^5}{5!} \\
&\quad - \cdots \\
&= \cdots
\end{aligned}$$

You will be relieved to know that you will not be expected to know how to do any of this more complicated kind of substitution.

3 Differentiation and integration

The next neat trick, which we have already seen some of, is differentiation and integration. The basic facts are these:

- $\displaystyle \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n x^n\right) = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$
- $\displaystyle \int_0^x \left(\sum_{n=0}^{\infty} c_n t^n\right) dt = \sum_{n=0}^{\infty} \int_0^x c_n t^n \, dt = \sum_{n=1}^{\infty} \frac{c_{n-1}}{n} x^n$
- The integral and derivative of a power series have the same radius of convergence as the power series itself.

Example 4. Find the Taylor series for $\cos x$.

Solution. $\square$

Example 5. Find the Taylor series for $\ln(1+x)$.

Solution. $\square$

Example 6. Find the Taylor series for $\int_0^x e^{-t^2}\, dt$.

Solution.

$\square$

This last example is particularly interesting because

$$f(x) = \int_0^x e^{-t^2}\, dt$$

is an extremely important function that *does not have a formula*—at least, not a finite formula. But clearly, it does have a very nice *infinite* formula.

Assignment 22: due Friday, May 25

Section 9.7, Problems 7–11 and 13–16. Problems 8, 10, 11, 14, and 16 will be graded carefully. Problem 11 will be discussed in tutorial on Thursday.

No class Monday

Monday, May 28 is Memorial Day.

Assignment 23: due Wednesday, May 30

To be decided. (Which I don't feel too guilty about this time, since it's not due until Wednesday.)

Math 133, Lecture 24: Algebraic operations on power series

Charles Staats

Friday, 25 May 2012

Exam

The final exam will be Friday, June 8, in Social Sciences 108, 10:30–12:30. Be there.

1 Important Taylor series

Here is a list of some important Taylor series with which you should be familiar. (It is borrowed, with modifications, from p. 495 of the textbook.)

(1)
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

(2)
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad -1 < x < 1$$

(3)
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad -1 < x \leq 1$$

(4)
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad -1 \leq x \leq 1$$

(5)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad -\infty < x < \infty$$

(6)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad -\infty < x < \infty$$

(7)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad -\infty < x < \infty$$

Note that you are not required to know, for the interval of convergence, whether the inequality sign is $<$ ($>$) or $\leq$ ($\geq$, respectively).

These may, perhaps, be remembered as follows:

(1) The geometric series: $1 + r + r^2 + r^3 + \cdots = 1/(1-r)$, as long as $|r| < 1$.

(2) Substitute $-x$ for x in (1).

(3) Integrate (2); remember $\int_0 \frac{dx}{1+x} = \ln(1+x)$.

(4) Substitute x^2 for x in (2), and then integrate. Remember $\int_0 \frac{dx}{1+x^2} = \arctan x$.

For (5)–(7), it may be best to remember the formula for general Taylor series about zero

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \end{aligned}$$

(which you should, in any case, know) and then remember the pattern of the derivatives at zero:

$f(x)$	$f(0)$	$f'(0)$	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$	$f^{(5)}(0)$	$\cdots$		
e^x	1	1	1	1	1	1	1	1	$\cdots$
$\sin x$	0	1	0	-1	0	1	0	-1	$\cdots$
$\cos x$	1	0	-1	0	1	0	-1	0	$\cdots$

No class Monday

Monday, May 28 is Memorial Day.

Assignment 23: due Wednesday, May 30

Section 9.6, Problems 5–8 and 15–16. Ignore the endpoints. Problems 6 and 16 will be graded carefully.

Section 9.8, Problems 1, 3, 4, 5, 7, 8, 10, 25, and 38. Problems 1, 4, 8, and 38 will be graded carefully. Problems 25 and 38 will be discussed in tutorial on Tuesday.

Section 9.9, Concepts Review 1 and 2.

Section 9.9, Problems 1–5. You may use your calculator once you have written down the polynomial. Problems 2 and 4 will be graded carefully.

Math 133, Lecture 25: What does the Taylor series converge to?

Charles Staats

Wednesday, 30 May 2012

Logistics

- The final exam will be Friday, June 8, in Social Sciences 108, 10:30–12:30. Be there.
- Friday's lecture will be a review session. Please come with questions.
- I will be holding extra office hours this afternoon (Wednesday), 1:30–2:30.

1 General Principles

Consider the following two general principles:

- There should be a "nice" function $f(t)$ such that the function is zero for $t \leq 0$ but nonzero for $t > 0$. This principle is derived from physical intuition: we expect there should be a "nice" function to describe the path of an object that starts off at rest, but then—say, after a push—begins to move.
- If a function is "nice," then its Taylor series converges to it in a neighborhood of every point where the function is defined.

The difficulty with these two principles is that they do not play well together. If we have a function $f(t)$ that is nice enough that all of its derivatives exist at every point, and $f(t) = 0$ for all $t \leq 0$, then $f'(t) = 0$ for all $t \leq 0$—and the same for f'', f''', $f^{(4)}$, etc. In particular,

$$f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = \cdots = 0.$$

Consequently, the Taylor series about zero is, quite simply,

$$0 + 0t + 0t^2 + 0t^3 + 0t^4 + 0t^5 + \cdots = 0.$$

If this Taylor series converges to f in a neighborhood of zero, as the second principle says it should, then $f(t) = 0$ on an open interval including zero; so, we cannot have $f(t) > 0$ for all $t > 0$, which appears to contradict the first principle.

In truth, perhaps surprisingly, the first principle is true and the second principle is false: there does exist such a function, and its Taylor series about zero does not, in fact, converge to it anywhere to the right of zero.

2 Example: The Taylor series for f can converge to something other than f

When we calculated, for instance, the Taylor series for $\sin x$,

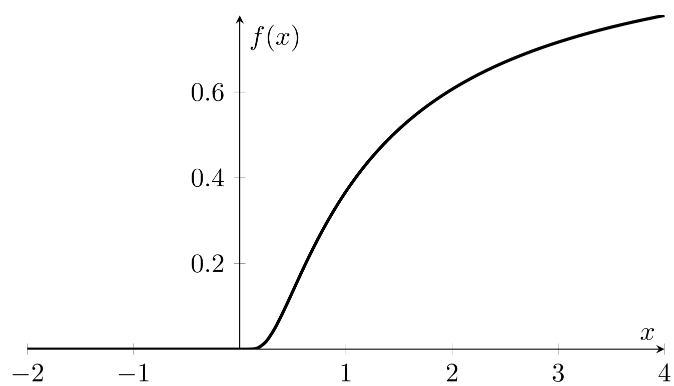
Exercise 1. What is the Taylor series for $\sin x$? (You should probably have this memorized.)

we then showed that the series has infinite radius of convergence; that is, the power series converges, absolutely, to *some* function. However, we did not actually show that this function was $\sin x$.

To see that this is, in fact, a question, consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

Here's the graph of f:



Some of you may recall my emphasis that functions need not be given by explicit formulas; and, in fact, that this is, to a large extent, the whole point of functions: sometimes, we need to study things that cannot be given by a single, explicit formula. In such a study, the function f that I have just defined is an example of an extremely important kind of function. It exhibits a couple of extremely important properties we have not seen before—at least, not together:

- All the derivatives (first, second, third, fourth, ...) exist at every point—just like for a function given by a single "nice" formula.

- The function is constant on part of its domain—but only part.

Physically, you might imagine this function as describing the path of an object that starts off stationary (not moving), but then, at some point, begins to move—without any sudden jerks (discontinuities in higher derivatives). And so, in some sense, you might expect such a function to exist; but we have not seen it until now.

On the other hand, mathematically, it presents something of a quandary. First, let's compute a the first and second derivatives at zero, just to convince you that they do exist.

By definition,

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h)}{h}. \end{aligned}$$

Given the way f is defined, we had best compute the left and right-hand limits separately.

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(h)}{h} &= \lim_{h\to 0^-} \frac{0}{h} = 0. \\ \lim_{h\to 0^+} \frac{f(h)}{h} &= \lim_{h\to 0^+} \frac{e^{-1/h}}{h} \\ &= \lim_{u\to\infty} \frac{e^{-u}}{u^{-1}} && \text{where } u = 1/h = h^{-1} \\ &= \lim_{u\to\infty} \frac{u}{e^u} && \left(\frac{\infty}{\infty}\right) \\ &= \lim_{u\to\infty} \frac{1}{e^u} \\ &= 0. \end{aligned}$$

Since both the one-sided limits are zero, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = 0.$$

Consequently, we have that

$$f'(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^{-2}e^{-1/x} & \text{if } x > 0. \end{cases}$$

Let's do the same to find $f''(0)$:

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{f'(0+h)-f'(0)}{h} \\ &= \lim_{h\to 0} \frac{f'(h)}{h}. \end{aligned}$$

Given the way f' is defined, we had best compute the left and right-hand limits separately.

$$\begin{aligned}
\lim_{h\to 0^-}\frac{f'(h)}{h} &= \lim_{h\to 0^-}\frac{0}{h} = 0. \\
\lim_{h\to 0^+}\frac{f'(h)}{h} &= \lim_{h\to 0^+}\frac{h^{-2}e^{-1/h}}{h} \\
&= \lim_{u\to\infty}\frac{u^2e^{-u}}{u^{-1}} && \text{where } u = 1/h = h^{-1} \\
&= \lim_{u\to\infty}\frac{u^3}{e^u} && \left(\frac{\infty}{\infty}\right) \\
&= 0.
\end{aligned}$$

Since both the one-sided limits are zero, we have

$$f''(0) = \lim_{h\to 0}\frac{f'(h)}{h} = 0.$$

Consequently, we have that

$$f''(x) = \begin{cases} 0 & \text{if } x \le 0, \\ (x^{-4} - 2x^{-3})e^{-1/x} & \text{if } x > 0. \end{cases}$$

It would be perhaps to complex to continue from here, but perhaps you will take my word that when we are evaluating the n^{th} derivative of f at 0, we will always end up looking at

$$\lim_{u\to\infty}\frac{\text{polynomial in } u}{e^u},$$

which is necessarily zero.

Thus, we end up with the following result:

$f^(n)(0) = 0$ for all n.

Exercise 2. (a) What is the Taylor series for f about zero?

(b) For what values of x does this series converge?

(c) For what values of x does this series converge *to* $f(x)$?

The first two questions have answers so simple you will wonder if you did something wrong. The third question is the interesting one; its answer is *not* the same as the answer to the second question.

3 The Taylor series for $\sin x$

On the other hand, the fact of the matter is that for most of the functions you encounter in this class, the Taylor series (if it is defined) will, in fact, converge to the function. There are a number of different ways to do this. Here is one way that sometimes works:

1. Find the Taylor series for f.
2. Show that the Taylor series converges to *some* function g. We want to show that $g = f$.
3. Find a differential equation that this Taylor series satisfies. Thus, g necessarily satisfies the differential equation.
4. Solve the differential equation to show that $g = f$.

You have already seen this once: we used this technique to show that e^x is equal to its Taylor series about zero. (In fact, this showed up on the last test, and may well show up on the final exam.) Perhaps now, you will have more appreciation for the technique.

Example 3. We already carried this procedure out for $f(x) = e^x$. The key step was, once we had found the Taylor series

$$g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots ,$$

to differentiate the Taylor series and note that we get exactly the same series back: $g'(x) = g(x)$. Setting $y = g(x)$, this shows that g satisfies the differential equation

$$\frac{dy}{dx} = y,$$

which we then solved to show that $g(x) = f(x)$.

Exercise 4. Let

$$g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

Show that this power series satisfies the differential equation

$$\frac{d^2y}{dx^2} = -y.$$

Solution.

(next page)

$\square$

Here is a trick to solve this differential equation (which you will not need to know for the test): Define a new variable/function

$$v = \frac{dy}{dx}.$$

(The v stands for "velocity," which makes sense sometimes.)

Then we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dv}{dx} \\
\frac{dv}{dy} &= \frac{dv}{dx}\cdot\frac{dx}{dy} && \text{chain rule} \\
&= \frac{dv}{dx}\cdot\frac{1}{dy/dx} \\
&= \frac{dv}{dx}\cdot\frac{1}{v} \\
&= \frac{d^2y}{dx^2}\cdot\frac{1}{v} \\
&= \frac{1}{v}\cdot(-y).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{dv}{dy} &= \frac{-y}{v} \\
\int v\ dv &= \int(-y)\ dy \\
\tfrac{1}{2}v^2 &= -\tfrac{1}{2}y^2 + C_1 \\
v^2 &= -y^2 + C_2 \\
\left(\frac{dy}{dx}\right)^2 &= C_2 - y^2 \\
\frac{dy}{dx} &= \pm\sqrt{C_2 - y^2} \\
\int\frac{dy}{\sqrt{C_2 - y^2}} &= \int \pm dx \\
\arcsin\left(\frac{y}{\sqrt{C_2}}\right) &= \pm x + C_3 \\
\frac{y}{\sqrt{C_2}} &= \sin(C_3 \pm x) \\
y &= \sqrt{C_2}\sin(C_3 \pm x) \\
&= \pm\sqrt{C_2}\sin(x \pm C_3) \\
&= A\sin(x - x_0),
\end{aligned}$$

where $A = \pm\sqrt{C_2}$ and $-x_0 = \pm C_3$.

$$\begin{aligned}
&= A\sin x\cos x_0 - Asinx_0\cos x \\
&= a\sin x + b\cos x,
\end{aligned}$$

where $a = A\cos x_0$ and $b = -A\sin x_0$. Thus,

$$g(x) = a\sin x + b\cos x$$

for some constants a and b. Now, let's bring in additional information we have about g to determine a and b. Let's restate the definition of g, which we defined as an infinite series:

$$\begin{aligned} g(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots & \qquad g(0) = 0 \\ g'(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots & \qquad g'(0) = 1 \end{aligned}$$

Thus,

$$0 = g(0) = a\sin 0 + b\cos 0 = a \cdot 0 + b \cdot 1 = b,$$

i.e., $b = 0$, so

$$\begin{aligned} g(x) &= a\sin x \\ g'(x) &= a\cos x \\ 1 = g'(0) &= a\cos 0 = a. \end{aligned}$$

Since $a = 1$, we conclude that

$$g(x) = \sin x,$$

i.e. that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots;$$

i.e., $\sin x$ is equal to the limit of its Taylor series. This is precisely what we were trying to show.