A SURPRISINGLY STRONG RELATIVE VERSION OF THE ALGEBRAIC HARTOGS' LEMMA

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Abstract. The Algebraic Hartogs' Lemma states that on a normal noetherian scheme, if a rational function is regular outside a closed subset of codimension at least two, then it can be uniquely extended to a global regular function. In this note, a relative version of this is proved, allowing the extension of a "flat family of rational sections" to a "family of regular sections." Surprisingly, two key conditions—that the schemes of the family be normal, and that the rational function be defined outside a subset of codimension at least two—are only required to hold on *generic* members of the family.

1. Some notes

The main result of this note is stated in Theorem 13. Anyone wishing to make use of it should also consult Example 3 for an important subtlety.

While this result is probably not original, it does not seem to appear in the literature; at least, the mathoverflow question [Sta], asking for such a reference, has so far not received one. A very similar result is given by Hassett and Kovács in Proposition 3.5 of [HK04]. Their result applies to coherent sheaves rather than just line bundles, but makes assumptions about every fiber that in this note are only required for the fibers over the associated points. It is possible that their proof techniques could give the stronger result.

Whether or not this result is original, it is at least less well-known than the author believes it should be. This motivates the current note, which is not intended for publication.

2. The algebra

Throughout this paper, let A denote a noetherian ring, and R a flat noetherian A-algebra. The set of all flat nonzerodivisors (Definition 1) will be denoted $S \subset R$.

Definition 1. A nonzerodivisor $s \in R$ will be called a *flat nonzerodivisor* if R/sR is flat over A.

Remark 1.1. The following are equivalent, for a nonzerodivisor $s \in R$:

(i) s is a flat nonzerodivisor.

(ii) For every A-module M,

$$R \otimes_A M \xrightarrow{s} R \otimes_A M$$

is injective.

(iii) s pulls back to a nonzerodivisor on every fiber of $\operatorname{Spec} R$ over $\operatorname{Spec} A$.

Date: September 8, 2012.

Remark 1.2. The set S of flat nonzerodivisors is multiplicatively closed, and in fact, saturated. To see this, note that S is the intersection of all the sets $R \smallsetminus \mathfrak{p}$, where $\mathfrak{p}$ ranges over the associated primes of R and of the fibers of $\operatorname{Spec} R$ over $\operatorname{Spec} A$.

Definition 2. For the purposes of this note, the ring homomorphism $A \to R$ is said to be *Hartogs* if the following holds:

Suppose that $\ell \in S^{-1}R$ is regular on an open set $U \subset \operatorname{Spec} R$ satisfying the following condition:

- For $y \in \operatorname{Spec} A$, let X_y be the fiber of $\operatorname{Spec} R \to \operatorname{Spec} A$ over y. Whenever y is an associated point of $\operatorname{Spec} A$, then $X_y \smallsetminus U$ has codimension at least two in X_y.

Then $\ell \in R$.

Remark 2.1. Let $I \subset R$ be the ideal of denominators of $\ell \in S^{-1}R$. Then $V(I) \subset \operatorname{Spec} R$ is precisely the minimal closed subset over which ℓ is not regular. Thus, the condition that "$X_y \smallsetminus U$ has codimension at least two in X_y" is equivalent to the condition that "the pullback of I to $R \otimes_A \Bbbk(y)$ has height at least two."

Remark 2.2. Note that the condition that "$X_y \smallsetminus U$ has codimension at least two in X_y" is a stronger restriction on ℓ than the condition that "ℓ pulls back to a rational function on X_y that is regular outside a set of codimension at least two." Even if $A \to R$ is Hartogs, rational functions $\ell \in S^{-1}R$ satisfying the second condition need not extend, as shown in the following example.

Example 3. Let $A = \Bbbk[x, \varepsilon]/(\varepsilon^2, x\varepsilon)$, and let $R = A[z]$. Proposition 12 will show that $A \to R$ is Hartogs.

Consider the rational function $\ell = \varepsilon/z \in S^{-1}R$. The ideal of denominators of ℓ is (x, ε, z); hence, ℓ is regular outside a subset of codimension 2. Furthermore, ℓ pulls back to the zero section in every fiber; clearly, the zero section is defined everywhere. However, $\ell \notin R$.

To see how ℓ fails the more restrictive hypothesis, consider the open set $U = \operatorname{Spec} R \smallsetminus \{(0,0)\}$ on which ℓ is regular *before* passing to fibers. Let $y \in \operatorname{Spec} A$ denote the embedded point corresponding to the ideal (x, ε) of A. Then $X_y \smallsetminus U$ has codimension 1 in X_y.

Lemma 4. If $\mathfrak{a}$ is an ideal of A, then $\mathfrak{a}S^{-1}R \cap R = \mathfrak{a}R$.

Proof. Let $\ell \in \mathfrak{a}S^{-1}R \cap R$. Then there exist $s \in S$ and $a \in \mathfrak{a}R$ such that $\ell = a/s$, i.e., $s\ell = a \in \mathfrak{a}R$. Consider the short exact sequence

$$0 \longrightarrow R \xrightarrow{s} R \longrightarrow R/sR \longrightarrow 0.$$

Since R/sR is A-flat, tensoring by $A/\mathfrak{a}$ gives a short exact sequence

$$0 \longrightarrow R/\mathfrak{a}R \xrightarrow{s} R/\mathfrak{a}R \longrightarrow R/(\mathfrak{a}R + sR) \longrightarrow 0,$$

which shows in particular that s is a nonzerodivisor on $R/\mathfrak{a}R$. Since $s\ell \in \mathfrak{a}R$, this implies $\ell \in \mathfrak{a}R$, as desired. □

Lemma 5. Let $a \in A$ and $\ell \in S^{-1}R/\operatorname{Ann}_{S^{-1}R}(a)$, where $S \subset R$ is as usual the set of flat nonzerodivisors in R. Let

- I be the ideal of denominators of ℓ over $R/\operatorname{Ann}(a)$, and let
- J be the ideal of denominators of $a\ell$ over R.

Then $J = I + \operatorname{Ann}_R(a)$.

Remark 5.1. $\operatorname{Ann}(a)$ is respected by flat extension, since, for instance,

$$\frac{R}{\operatorname{Ann}_A(a)R} \xhookrightarrow{a} R.$$

Thus, $\operatorname{Ann}_R(a) = \operatorname{Ann}_A(a)R$ and $\operatorname{Ann}_{S^{-1}R}(a) = \operatorname{Ann}_A(a)S^{-1}R$.

Proof of Lemma 5. (1) $J \supset I + \operatorname{Ann}_R(a)$. Suppose $s \in I + \operatorname{Ann}_R(a)$. Then $\bar{s}\ell \in R/\operatorname{Ann}(a)$ holds inside $S^{-1}R/\operatorname{Ann}(a)$. Hence $s\ell \in R + \operatorname{Ann}_{S^{-1}R}(a)$. Multiplying by a gives that $sa\ell \in R$.

(2) *It suffices to show that the sequence of R-modules*

$$0 \longrightarrow \frac{S^{-1}R/\operatorname{Ann}_{S^{-1}R}(a)}{R/\operatorname{Ann}_R(a)} \xrightarrow{a} \frac{S^{-1}R}{R} \longrightarrow \frac{S^{-1}R}{R + aS^{-1}R} \longrightarrow 0 \tag{6}$$

is exact. Suppose $s \in J$. Then $sa\ell \in R$. The goal is to show that $s\ell \in R/\operatorname{Ann}(a)$. It suffices to show that

$$\frac{S^{-1}R/\operatorname{Ann}_{S^{-1}R}(a)}{R/\operatorname{Ann}_R(a)} \xrightarrow{a} \frac{S^{-1}R}{R}$$

is injective, which is equivalent to the exactness of (6).

(3) *It suffices to show that* $\operatorname{Tor}_1^A\left(\frac{A}{aA}, \frac{S^{-1}R}{R}\right) = 0$. The sequence (6) is obtained by applying $\otimes_A \frac{S^{-1}R}{R}$ to the short exact sequence of A-modules

$$0 \longrightarrow \frac{A}{\operatorname{Ann}(a)} \xrightarrow{a} A \longrightarrow \frac{A}{aA} \longrightarrow 0.$$

(4) Consider the short exact sequence

$$0 \longrightarrow R \longrightarrow S^{-1}R \longrightarrow \frac{S^{-1}R}{R} \longrightarrow 0$$

as a sequence of A-modules. Tensoring by A/aA gives the long exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^A\left(\frac{A}{aA}, \frac{S^{-1}R}{R}\right) \longrightarrow \frac{R}{aR} \xrightarrow{\varphi} \frac{S^{-1}R}{aS^{-1}R} \longrightarrow \frac{S^{-1}R}{R + aS^{-1}R} \longrightarrow 0.$$

It follows by Lemma 4 that $aS^{-1}R \cap R = aR$. Thus, φ is injective, and so $\operatorname{Tor}_1^A\left(\frac{A}{aA}, \frac{S^{-1}R}{R}\right) = 0$, as desired. □

Lemma 7. If $(A, \mathfrak{n})$ is a local Artinian ring and the generic fiber $A/\mathfrak{n} \to R/\mathfrak{n}R$ is Hartogs, then $A \to R$ is Hartogs.

Proof. (1) $\mathfrak{n}$ *is the nilradical of A.*

(2) *Induct on* $\operatorname{length}_A(A)$. If $\operatorname{length}_A(A) = 1$, then $\mathfrak{n} = 0$, and so $A \to R$ is Hartogs by hypothesis.

(3) *There exists $\varepsilon \in \mathfrak{n}$ such that* $\mathfrak{n} = \operatorname{Ann}(\varepsilon)$.

Let $\ell \in S^{-1}R$, with ideal of denominators J such that $\operatorname{ht}(J) \geq 2$. (Note that the height of J in R is equal to the height of $J/\mathfrak{n}$ in $R/\mathfrak{n}$.) It suffices to show that $\ell \in R$.

(4) *Without loss of generality, $\ell \in \varepsilon S^{-1}R$.* By the induction hypothesis, $A/(\varepsilon) \to R/\varepsilon R$ is Hartogs, so ℓ is regular modulo $\varepsilon S^{-1}R$. Subtracting an element of R from ℓ does not change its ideal of denominators.

(5) *There exists a unique $\ell' \in S^{-1}R/\mathfrak{n}S^{-1}R$ such that $\varepsilon\ell' = \ell$.* By choice of ε, $\operatorname{Ann}(\varepsilon) = \mathfrak{n}$.

(6) *The ideal of denominators of* ℓ' *has height at least* 2. Apply Lemma 5.

(7) $\ell' \in R/\mathfrak{n}R$. By the previous step, ℓ' is regular outside a closed subset of codimension at least two. Use the fact that $A/\mathfrak{n} \to R/\mathfrak{n}R$ is Hartogs.

Thus, $\ell = \varepsilon\ell' \in R$, as desired. ☐

In the proof of the following lemma, point 3 was suggested by Will Sawin in [Saw].

Lemma 8. If A is a coprimary ring (i.e., every zero divisor is nilpotent), and the fiber of $\operatorname{Spec} R$ over the generic point[1] of $\operatorname{Spec} A$ is Hartogs, then $A \to R$ is Hartogs.

Proof. Let K denote the total ring of fractions of A, so that K is an Artinian local ring. Consider the rings A, K, R, $R \otimes_A K$, and $S^{-1}R$.

(1) *The rings listed above are naturally identified with subrings of the total ring of fractions of* R. The inclusions fit into the following diagram, where $\widetilde{R}$ denotes the total ring of fractions of R:

$$\begin{array}{ccccccc} K & \hookrightarrow & R \otimes_A K & \hookrightarrow & S^{-1}R \otimes_A K & \searrow & \\ \uparrow & & \uparrow & & \uparrow & & \widetilde{R} \\ A & \hookrightarrow & R & \hookrightarrow & S^{-1}R & \nearrow & \end{array} \tag{9}$$

The injectivity of $K \to R \otimes_A K$ may be shown by induction on the length of K, via the diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K/\operatorname{Ann}(\varepsilon) & \xrightarrow{\varepsilon} & K & \longrightarrow & K/(\varepsilon) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R \otimes_A K/\operatorname{Ann}(\varepsilon) & \xrightarrow{\varepsilon} & R \otimes_A K & \longrightarrow & R \otimes_A K/(\varepsilon) & \longrightarrow & 0 \end{array}$$

where $\varepsilon \in K$ is any zero divisor. The injectivity of $A \to R$ follows from the commutativity of (9).

(2) $K \to R \otimes_A K$ *is Hartogs.* Use Lemma 7.

(3) *It suffices to show that* $(R \otimes_A K) \cap S^{-1}R = R$. Use the definition of *Hartogs.*

Thus, let $\ell \in (R \otimes_A K) \cap S^{-1}R$. Since $\ell \in R \otimes_A K$, necessarily $\ell = f/a$, for some $f \in R$ and some nonzerodivisor $a \in A$. Thus,

$$f = a\ell \in aS^{-1}R.$$

Since $f \in R$ by hypothesis, Lemma 4 shows that $f \in aR$. Hence, $\ell = f/a \in R$, as desired. ☐

The following lemma, and its use in a proof similar to the proof for Lemma 11, was suggested by Will Sawin in [Saw].

Lemma 10. (Glueing over closed subsets) Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in A. Then the natural sequence of A-modules

$$0 \longrightarrow \frac{A}{\mathfrak{a} \cap \mathfrak{b}} \longrightarrow \frac{A}{\mathfrak{a}} \oplus \frac{A}{\mathfrak{b}} \longrightarrow \frac{A}{\mathfrak{a} + \mathfrak{b}}$$

[1] Another characterization of a coprimary ring is that it has a unique associated prime, which I have called the "generic point" of $\operatorname{Spec} A$.

is exact. (The last map is the difference of the two residues.)

Proof. (1) This is equivalent to the following statement: If $f, g \in A$ are such that $f \equiv g \pmod{\mathfrak{a}+\mathfrak{b}}$, then there exists $\alpha \in A$ such that $\alpha \equiv f \pmod{\mathfrak{a}}$ and $\alpha \equiv g \pmod{\mathfrak{b}}$.

(2) It is given that $f - g = a - b$, for some $a \in \mathfrak{a}$ and some $b \in \mathfrak{b}$. Let $\alpha = f - a = g - b$. □

Lemma 11. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in A. If $A/\mathfrak{a} \to R/\mathfrak{a}R$ and $A/\mathfrak{b} \to R/\mathfrak{b}R$ are both Hartogs, then so is $A/(\mathfrak{a}\cap\mathfrak{b}) \to R/(\mathfrak{a}\cap\mathfrak{b})R$.

Proof. (1) *It may be assumed that neither of* $\mathfrak{a}$, $\mathfrak{b}$ *is contained in the other.* Otherwise, the statement is obvious.

(2) *The associated primes of* $A/(\mathfrak{a}\cap\mathfrak{b})$ *are among those primes associated to at least one of* $A/\mathfrak{a}$, $A/\mathfrak{b}$. A primary decomposition of $\mathfrak{a}$ together with a primary decomposition of $\mathfrak{b}$ give a primary decomposition of $\mathfrak{a}\cap\mathfrak{b}$.

(3) *Without loss of generality,* $\mathfrak{a}\cap\mathfrak{b} = 0$.

Let $S \subset R$ be the set of all flat nonzerodivisors over A. Let $\ell \in S^{-1}R$ be regular on an open subset $U \subset \operatorname{Spec} R$ that contains all but a codimension-two subset of each associated fiber. The goal is to show that $\ell \in R$.

(4) *There exist* $r, r' \in R$ *such that*

$$r \equiv \ell \pmod{\mathfrak{a}S^{-1}R} \quad \textit{and} \quad r' \equiv \ell \pmod{\mathfrak{b}S^{-1}R}.$$

Consequently,

$$r \equiv r' \pmod{(\mathfrak{a}+\mathfrak{b})S^{-1}R}.$$

(5) $r - r' \in (\mathfrak{a}+\mathfrak{b})R$. Apply Lemma 4.

(6) $\ell \in R$. Tensor the exact sequence of Lemma 10 by the flat A-module R. □

Proposition 12. If every associated fiber of $\operatorname{Spec} R \to \operatorname{Spec} A$ is normal, then $A \to R$ is Hartogs.

Proof. Take a primary decomposition of the zero ideal in A. By Lemma 8, whenever $\mathfrak{p}$ is an associated prime of A and $\mathfrak{a}$ is $\mathfrak{p}$-primary, $A/\mathfrak{a} \to R/\mathfrak{a}R$ is Hartogs. Apply Lemma 11. □

3. Moving to a scheme-theoretic statement

Theorem 13. Let $X \to Y$ be a flat morphism of noetherian schemes such that every associated fiber is normal. Let $\mathcal{L}$ be a line bundle on X. Suppose that $i\colon U \hookrightarrow X$ is an open subscheme such that

(i) U contains all the associated points of X,
(ii) for every $y \in Y$, $U \cap X_y$ contains all the associated points of X_y, and
(iii) for every associated point η of Y, U contains all but a codimension-two closed subset of X_η.

Then the natural sheaf homomorphism

$$r\colon \mathcal{L} \longrightarrow i_*(\mathcal{L}|U)$$

is an isomorphism.

Proof. (1) It suffices to show that the theorem holds locally. More precisely, it suffices to show the theorem holds when $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} A$, and $\mathcal{L} \cong \mathcal{O}_X$, with $(A, \mathfrak{p}) \to (R, \mathfrak{m})$ being a local homomorphism of local rings.

(2) Injectivity of r follows from the fact that U contains all the associated points of X.

(3) *If* $f \in \Gamma(U, \mathcal{O}_X)$*, then* $f = g/s$*, for some* $g \in R$ *and some flat nonzerodivisor* $s \in R$*.* Let $I \subset R$ be the ideal of denominators for f, so that $V(I) \subset X \smallsetminus U$. By prime avoidance, there exists $s \in I$ that does not lie in any associated prime of R or of $R/\mathfrak{p}R$. Hence, s is a nonzerodivisor in R and in $R/\mathfrak{p}R$. By the local criterion for flatness, $R/(s)$ is flat over A.

(4) *The injection* $R \hookrightarrow \Gamma(U, \mathcal{O}_X)$ *is an isomorphism.* Apply Proposition 12. □

References

[HK04] Brendan Hassett and Sándor J. Kovács. Reflexive pull-backs and base extension. *J. Algebraic Geom.*, 13(2):233–247, 2004.

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