Math 131, Lecture 19: The Chain Rule

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Important Note: There have been a few changes to Assignment 17. Don't use the version from Lecture 18.

1 Notes on the quiz

I've finally graded the quiz, and I wanted to make a few remarks.

1. Even apart from the one massive typo, the quiz directions were confusing. I apologize for this. Next time (tomorrow), I will make sure to take more time and care in designing the quiz.

   Because the instructions were confusing, the quiz was difficult to grade. Since the grades don't actually count for anything, I did not try too hard to assign grades based on what I thought was a "fair" assessment of your work. Instead, I mostly concentrated on assigning grades—and making comments—in whatever way I thought would be most useful to you.

2. In particular, on the delta-epsilon proof, assigning "fair" scores would have been virtually impossible since the statement of the problem was incorrect. Among other things, I deducted points for clear problems in the style of the δ-ε proofs.

3. If δs and εs should appear in your solution, I will say so explicitly in the problem instructions. In particular, if I ask you for the "limit-based definition of the derivative," I'm looking for something like

   $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

   If I don't ask you for εs and δs, don't give them to me.

4. Do not—EVER—follow an expression like $\lim_{h \to 0}$ by an equals sign:

   $$\xcancel{f'(x) = \lim_{h \to 0} = \frac{f(x+h) - f(x)}{h}}$$

This simply makes no sense. The notation $\lim_{h\to 0}$ is supposed to represent the limit *of* an expression. If you instead follow it by an equals sign, it's like saying "The limit of is. . . ."

If you write this on a test, I will deduct points.

5. When I ask you to give an ε-δ proof for a limit of a piecewise-linear function, my secret goal is to get you to understand how you might go about proving the following statement:

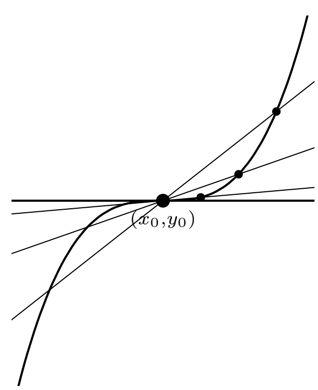
   > The two-sided limit exists and equals ℓ if and only if both the one-sided limits exist and equal ℓ.

   Some of you gave separate ε-δ proofs of both the one-sided limits, and then used the fact above to deduce the value of the two-sided limit. This is perfectly correct, and would have received full credit if I were grading the quiz for credit. However, since I was grading primarily to let you know whether you are prepared for this sort of problem on the test, I deducted a couple points. I very probably will give a problem like this on the test; if I do so, I will explicitly state that you are not allowed to use the statement above, so that I can justifiably deduct points if you do.

   Having reread the sentence above, I realized it sounds like I am looking for excuses to deduct points. This is NOT the case. When I give a particular problem, there are certain things I am trying to see if you understand. If you can do the problem correctly without understanding these things, then my whole purpose in giving the problem is compromised.

2 Computing the derivative from the definition

There are several limit-based definitions of the derivative. They all amount to saying "take the limit of the slope of the secant line between two points, as those two points get close together:"



The easiest definition to remember is probably

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

The easiest to compute with is probably based on the so-called "difference quotient":

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If I ask you to compute a derivative from the definition, I'm not mostly interested in whether you can find the answer. I may even phrase the question something like the following[1]:

> Use the definition of the derivative to prove that if f is the function defined by $f(x) = 1/x$, then $f'(x) = -1/x^2$.

As you may note, I am in fact giving you the "answer" here: $f'(x) = -1/x^2$. What I care about, when I ask a question like this, is whether you understand the definition well enough to use it.

Although we've done this example in Lecture 17, I'm going to repeat it here, since I will need it later in the lecture.

Example 1. Use the definition of the derivative to prove that if f is the function defined by $f(x) = \frac{1}{x}$, then $f'(x) = \frac{-1}{x^2}$.

Solution.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)} \\
&= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x^2}.
\end{aligned}$$

□

If you want more examples of this sort of computation, you should review Lecture 17. (There's a version on Chalk that includes solutions to all of the examples.)

[1] NOTE: If I ask you this question on the test, the function f will be different.

3 Differentiating Quotients

We can use the Chain Rule together with the Product Rule and Example 1 (page 3) to differentiate quotients.

Example 2. Find $\frac{d}{dx}\left(\frac{1}{x-1}\right)$.

Recall, from Example 1, that $D_x(1/x) = -1/x^2$.

Solution.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x-1}\right) &= \frac{-1}{(x-1)^2}\cdot\frac{d}{dx}(x-1)\\ &= \frac{-1}{(x-1)^2}.\end{aligned}$$

□

4 Proof Sketch of the Chain Rule

Let y be a function of x. What does it mean to say that the derivative of y at x_0 is equal to a number m?

$$\begin{aligned}&\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = m\\ \iff& \forall\varepsilon > 0,\ \exists\delta > 0 \text{ s.t. if } 0 < |\Delta x| < \delta, \text{ then } \left|\frac{\Delta y}{\Delta x} - m\right| < \varepsilon\\ \iff& \forall\varepsilon > 0,\ \exists\delta > 0 \text{ s.t. if } 0 < |\Delta x| < \delta, \text{ then } \frac{|\Delta y - m\Delta x|}{|\Delta x|} < \varepsilon\\ \iff& \forall\varepsilon > 0,\ \exists\delta > 0 \text{ s.t. if } 0 < |\Delta x| < \delta, \text{ then } |\Delta y - m\Delta x| < \varepsilon|\Delta x|\end{aligned}$$

Informally, this means that the statement $m = f'(x)$ is equivalent to the statement that

If the change in x is small, then $\Delta y \approx m\Delta x$.

In other words, near x_0, y is approximated by the tangent line. See Figure 1.

This suggests a (non-rigorous) definition of the derivative using infinitesimals: if $y = f(x)$, then $f'(x)$ is the number such that

$$dy = f'(x)\,dx.$$

This "definition" is based on the general notion that "if something is approximately true for small Δx, then it should be exactly true for dx because dx is so small." Thus, since $\Delta y \approx f'(x)\Delta x$, we get $dy = f'(x)\,dx$. This principle can get you in big trouble if applied indiscriminately, which is why using infinitesimals

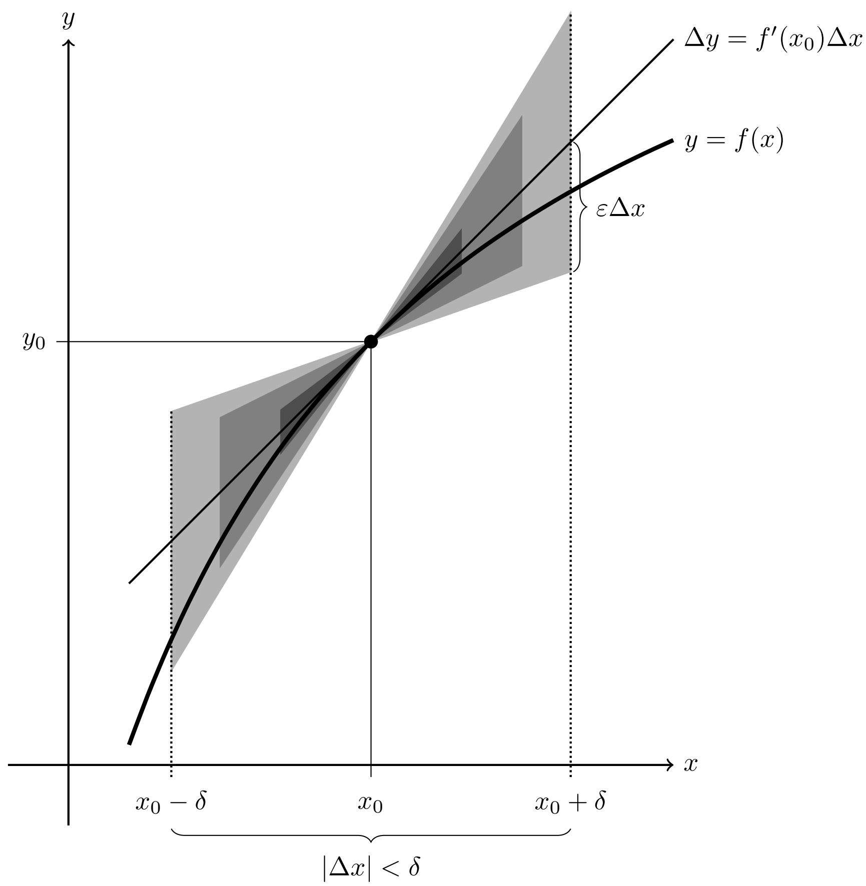


Figure 1: When Δx is small, then Δy is approximated by $f'(x_0)\Delta x$. In other words, for Δx small, the function is approximated by its tangent line (which is defined by $\Delta y = f'(x_0)\Delta x$). More precisely, the function is contained in a narrow cone about the tangent line. The width of the cone is controlled by ε. We can make the cone as narrow as we want ("arbitrarily narrow"), by making δ (and hence Δx) sufficiently small.

is "walking on clouds." But in many circumstances, it can give good intuition and correct results.

Now, suppose that $y = f(u)$ and $u = g(x)$, so that $y = f(u) = f(g(x))$. Then we have $dy = f'(u)\,du$ and $du = g'(x)\,dx$, so

$$\begin{aligned} dy &= f'(u)\,du \\ &= f'(u)g'(x)\,dx \\ &= f'(g(x))g'(x)\,dx. \end{aligned}$$

Hence, by the "infinitesimal definition of the derivative,"

$$\frac{dy}{dx} = f'(g(x))g'(x).$$

> Note: If I ask you on a test for the "Leibniz derivation of the Chain Rule" or the "Infinitesimal derivation of the Chain Rule," I am asking you, more or less, to give me the paragraph above.

Theorem. (Chain Rule) If f and g are differentiable functions, then $f \circ g$ is also differentiable, and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

The proof of the Chain Rule is to use εs and δs to say exactly what is meant by "approximately equal" in the argument

$$\begin{aligned} \Delta y &\approx f'(u)\Delta u \\ &\approx f'(u)g'(x)\Delta x \\ &= f'(g(x))g'(x)\Delta x. \end{aligned}$$

Unfortunately, there are two complications that have to be dealt with. The first is that, for technical reasons, we need an ε-δ definition for the derivative that allows $|\Delta x| = 0$. The following statement turns out to work:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. if } |\Delta x| < \delta, \text{ then } |\Delta y - f'(x_0)\Delta x| \leq \varepsilon|\Delta x|.$$

Comparing this to the earlier version, we got rid if the requirement $0 < |\Delta x|$ by changing the final $< \varepsilon|\Delta x|$ to $\leq \varepsilon|\Delta x|$. I don't want to explain why exactly we can do this, but anyone who has taken (and understood) an analysis course ought to be able to do it without much trouble.

The second complication is that the expression for δ in terms of ε turns out to be a bit ugly. For this reason, I will spare you the details. However, I hope I have convinced you that the basic idea of the proof of the Chain Rule is comprehensible, even if the technical details are a bit involved.

Assignment 17 (due Friday, 11 November)

From Section 2.3:

- Problems 5–8. Do each problem two ways—using the limit definition of your choice, and using the rules of differentiation (including the Chain Rule, if you find it helpful).
- Problems 17–20.
- Problems 31–32. Do not FOIL out the products; instead, use the product rule for differentiation.

The even-numbered problems will be graded carefully.

Section 2.5, Problems 1–4. Make sure it is clear, from your answer, how you are using the Chain Rule (see, for instance, Example 3 at the end of Lecture 18). Problems 2 and 4 will be graded carefully.

Give an ε-δ proof for each of the following. Do *not* use the fact that if both the one-sided limits exist and are equal, then the two-sided limit exists and is equal to both of them.

1. Let f be the function defined by

$$f(x) = \begin{cases} 7x - 3 & \text{if } x \leq 0, \\ -\frac{1}{9}x - 3 & \text{if } x > 0. \end{cases}$$

Show that $\lim_{x \to 0} f(x) = -3$.

2. Let f be the function defined by

$$f(x) = \begin{cases} -\frac{1}{7}x - \frac{18}{7} & \text{if } x < 3, \\ \frac{1}{6}x - \frac{7}{2} & \text{if } x \geq 3. \end{cases}$$

Show that $\lim_{x \to 3} f(x) = -3$.

3. Let f be the function defined by

$$f(x) = \begin{cases} -\frac{1}{2}x + \frac{3}{2} & \text{if } x < -3, \\ 4 & \text{if } x = -3, \\ 3x + 12 & \text{if } x > -3. \end{cases}$$

Show that $\lim_{x \to -3} f(x) = 3$.

Problems 1 and 3 will be graded carefully.

[NOTE: This is now a Bonus Problem.] Suppose $y = f(x)$ and $f(x_0) = y_0$. A purely ε-δ version of the statement that "f is continuous at x_0" is given as follows:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. if } |x - x_0| < \delta, \text{ then } |y - y_0| < \varepsilon.$$

Use this definition to prove the following fact:

> Suppose that
>
> $$\begin{aligned} u &= f(x), \\ u_0 &= f(x_0), \\ y &= g(u) = g(f(x)), \quad \text{and} \\ y_0 &= g(u_0) = g(f(x_0)). \end{aligned}$$
>
> If f is continuous at x_0 and g is continuous at u_0, then $g \circ f$ is continuous at x_0.

Assignment 18 (due Monday, 14 November

Section 2.2, Problems 9, 10, 15, and 16. These problems are about the process of computing the derivative from the limit; finding the "answer" by another method will not receive full credit. You do not need to hand in any of these, but a similar problem will appear on the test.

Section 2.3, Problems 27–30. Use the Product Rule. Problems 28 and 30 will be graded carefully.

Section 2.5, Problems 5–8, 13–14, and 17–18. You do not need to show every single step, but it should be clear to the grader how you got to the answer. The even-numbered problems will be graded carefully.

Differentiate the following expressions with respect to x. (Hint: Apply the Chain Rule more than once.) You do not need to show every single step, but it should be clear to the grader how you got to the answer. You do not need to simplify the answer.

1. $\left(5(2x+1)^{361} - 17\right)^{42}$
2. $\left(1 - (1-2x)^{33}\right)^{1776}$

Both of these will be graded carefully. Since a similar problem may appear on the test, and this homework set will almost certainly not be graded before the test, you may want to ask to go over the answers in tutorial on Tuesday.

Give ε-δ proofs of the following facts. Do *not* use the fact that if both the one-sided limits exist and are equal, then the two-sided limit exists and is equal to both of them.

1. Let f be the function defined by

$$f(x) = \begin{cases} -8x + 10 & \text{if } x \leq 1, \\ 3x - 1 & \text{if } x > 1. \end{cases}$$

   Show that $\lim_{x \to 1} f(x) = 2$.

2. Let f be the function defined by

$$f(x) = \begin{cases} -4x + 5 & \text{if } x < 1, \\ -\frac{1}{2}x + \frac{3}{2} & \text{if } x > 1. \end{cases}$$

   Show that $\lim_{x \to 1} f(x) = 1$.

3. Let f be the function defined by

$$f(x) = \begin{cases} -8x - 2 & \text{if } x < 0, \\ -2 & \text{if } x = 0, \\ \frac{1}{7}x - 2 & \text{if } x > 0. \end{cases}$$

   Show that $\lim_{x \to 0} f(x) = -2$.

Problems 1 and 3 will be graded carefully.

Test II around Wednesday, 16 November

Experienced teachers of Math 131 tell me that you will probably have a lot of papers and the like due around the time of the test, and consequently will not have a lot of time to study for it. Thus, I suggest you start studying now. You may also want to think in terms of "practicing" rather than "studying": redoing old quiz and homework problems (without looking at the solutions, if you have them, until afterwards) may be more helpful than simply reading over them.